

Pin⁻(2)-MONOPOLE INVARIANTS

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ABSTRACT. We introduce a diffeomorphism invariant of 4-manifolds, Pin⁻(2)-monopole invariant, defined by using Pin⁻(2)-monopole equations. We compute the invariants of several 4-manifolds, and prove gluing formulae for these invariants. By using the invariants, we construct exotic smooth structures on $E(n) \# (\#_{i=1}^k (\Sigma_1^{(i)} \times \Sigma_2^{(i)}))$ such that $g(\Sigma_1^{(i)}) = 0, 1$ and $g(\Sigma_2^{(i)}) > 0$. As another application, we give an estimate of the genus of surfaces embedded in a 4-manifold X representing a class $\alpha \in H_2(X; l)$, where l is a local coefficient on X .

1. Introduction

In the paper [17], we introduced the Pin⁻(2)-monopole equations which are a twisted or a real version of the Seiberg-Witten equations, and obtained several constraints on the intersection forms with local coefficients of 4-manifolds by analyzing the moduli spaces. In this article, we investigate diffeomorphism invariants defined by using the Pin⁻(2)-monopole equations, which we will call Pin⁻(2)-*monopole invariants*. We compute the invariants of several 4-manifolds, and prove connected-sum formulae for these. We give two applications of these invariants. The first application is to construct exotic smooth structures on $E(n) \# (\#_{i=1}^k (\Sigma_1^{(i)} \times \Sigma_2^{(i)}))$ such that $g(\Sigma_1^{(i)}) = 0, 1$ and $g(\Sigma_2^{(i)}) > 0$. The second application is an estimate of the genus of surfaces embedded in a 4-manifold X representing a class $\alpha \in H_2(X; l)$, where l is a local coefficient on X , which can be considered as a local coefficient analogue of the adjunction inequality in the Seiberg-Witten theory [12, 5, 15, 20]. First, we state the applications.

1(i). **Exotic smooth structures.** Here is the first application:

Theorem 1.1. *For any positive integer n , there exists a set \mathcal{S}_n of infinitely many distinct smooth structures on the elliptic surface $E(n)$ which have the following significance: For $\sigma \in \mathcal{S}_n$, let $E(n)_\sigma$ be the manifold with the smooth structure σ homeomorphic to $E(n)$. For each positive integer i , let $\Sigma_1^{(i)} \times \Sigma_2^{(i)}$ be the direct product of two closed Riemann surfaces such that the genus of $\Sigma_1^{(i)}$ is 0 or 1 and that of $\Sigma_2^{(i)}$ is positive. Then, for any positive integer k , $E(n)_\sigma \# (\#_{i=1}^k (\Sigma_1^{(i)} \times \Sigma_2^{(i)}))$ for different σ are mutually non-diffeomorphic.*

Remark 1.2. A famous result due to C. T. C. Wall tells us that any pair of simply-connected smooth 4-manifolds M_1 and M_2 which have isomorphic intersection forms are *stably diffeomorphic* for stabilization by taking connected sums with $k(S^2 \times S^2)$ for sufficiently large

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k . (See e.g. [10].) Theorem 1.1 says that there exist infinitely many exotic structures on $E(n)$ which can not be stabilized by $\Sigma_1 \times \Sigma_2$ such that $g(\Sigma_1) = 0, 1$ and $g(\Sigma_2) > 0$.

1(ii). **The genus of embedded surfaces.** Let X be a closed oriented connected 4-manifold and suppose a nontrivial double covering $\tilde{X} \rightarrow X$ is given, and let $l = \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$. Then a homology class $\alpha \in H_2(X; l)$ is represented by an embedded surface Σ as follows:

- Σ is a connected surface embedded in X . Let $i: \Sigma \rightarrow X$ be the embedding map.
- The orientation system of Σ is identified with the pull-back i^*l of l by i .
- If $i_*: H_2(\Sigma; i^*l) \rightarrow H_2(X; l)$ is the induced homomorphism and $[\Sigma] \in H_2(\Sigma; i^*l)$ is the fundamental class, then $\alpha = i_*[\Sigma]$.

Conversely, a connected embedded surface Σ whose orientation system is the restriction of l has its fundamental class $[\Sigma]$ in $H_2(X; l)$.

Theorem 1.3. *Suppose a pair (X, l) of 4-manifold X , and a \mathbb{Z} -bundle l over X is one of the following:*

- $(N_1 \# N_2 \# \cdots \# N_n, l_1 \# \cdots \# l_n)$, where each N_i is a homotopy Enriques surface, and l_i is a nontrivial \mathbb{Z} -bundle, or
- $(K \# (\#_{i=1}^k (\Sigma_1^{(i)} \times \Sigma_2^{(i)})), l)$ in Theorem 1.1 with K a K3 surface, where $l = \underline{\mathbb{Z}} \# l_{X_2}$ is defined before Theorem 1.12.

Let Σ be a connected embedded surface as above representing a class $\alpha \in H_2(X; l)$. If α has an infinite order and $\alpha \cdot \alpha \geq 0$, then

$$-\chi(\Sigma) \geq \alpha \cdot \alpha.$$

Remark 1.4. The number $\alpha \cdot \alpha$ is the normal Euler number of the embedding $\Sigma \subset X$.

From this, we can also obtain some kind of equivariant adjunction inequality on the double coverings:

Corollary 1.5. *Let $\tilde{X} \rightarrow X$ be the double covering associated with (X, l) in Theorem 1.3, and $\iota: \tilde{X} \rightarrow \tilde{X}$ be the covering transformation. Suppose an oriented connected surface Σ embedded in \tilde{X} satisfies the property that $[\Sigma] - \iota_*[\Sigma]$ has an infinite order in $H_2(\tilde{X}; \mathbb{Z})$ and $[\Sigma] \cdot [\Sigma] \geq 0$. If $\Sigma \cap \iota(\Sigma) = \emptyset$, then*

$$(1.6) \quad -\chi(\Sigma) \geq [\Sigma] \cdot [\Sigma].$$

Example 1.7. Let us examine Corollary 1.5 for a simple example. Let $X = K3 \# (T^2 \times S^2)$. Then \tilde{X} is associated to a nontrivial double cover $T^2 \times S^2 \rightarrow T^2 \times S^2$ and hence $\tilde{X} = K_1 \# (T^2 \times S^2) \# K_2$, where K_i are copies of K3. Let $\sigma = [pt \times S^2]$ and $\tau = [T^2 \times pt]$ in $H_2(T^2 \times S^2; \mathbb{Z})$. Take a 2-sphere S representing σ embedded in the $T^2 \times S^2$ -component, and oriented connected surfaces Σ_i ($i = 1, 2$) embedded in the K_i -components so that $[\Sigma_i] \neq 0$, $[\Sigma_i]^2 \geq 0$, $\iota(\Sigma_1) \cap \Sigma_2 = \emptyset$ and $\iota_*[\Sigma_1] \neq [\Sigma_2]$. Then we can arrange to take a connected sum $\Sigma = \Sigma_1 \# S \# \Sigma_2$ in \tilde{X} such that $\Sigma \cap \iota(\Sigma) = \emptyset$. Such a Σ certainly satisfies (1.6) because of the adjunction inequality for K3. On the other hand, we can construct oriented connected surfaces Σ embedded in \tilde{X} with $\Sigma \cap \iota(\Sigma) \neq \emptyset$ which violate (1.6) as follows. Let g_1 be the genus of Σ_1 above. We can take an embedded 2-torus T representing

$\tau + n\sigma$ so that $2n > 2g_1 - [\Sigma_1]^2$. Then take a connected sum $\Sigma = \Sigma_1 \# T$ in \tilde{X} . Since $[\Sigma] \cdot \iota_*[\Sigma] = (\sigma + n\tau)^2 = 2n > 0$, we have $\Sigma \cap \iota(\Sigma) \neq \emptyset$.

1(iii). **Pin⁻(2)-monopole invariants.** To prove the results above, a Pin⁻(2)-monopole version of the Seiberg-Witten invariants will be defined and used. We remark that the Pin⁻(2)-monopole equations are defined on Spin^c-structures (§2(i) and [17], Section 3), which is a Pin⁻(2)-analogue of Spin^c-structures. (A natural view point is that a Spin^c-structure is assumed as an object defined on a double covering $\tilde{X} \rightarrow X$ of a manifold X . See §2(i).) Therefore these invariants can be considered as functions on Spin^c-structures. One of the special feature of the Pin⁻(2)-monopole theory is that the moduli spaces may be nonorientable. Hence, in general, \mathbb{Z}_2 -valued invariants will be defined. Only when the moduli space is orientable, \mathbb{Z} -valued invariants can be defined. Here, we state several nonvanishing results on the Pin⁻(2)-monopole invariants.

If N_0 be an Enriques surface, then there is a double covering $\pi: K_0 \rightarrow N_0$ with K_0 a K3 surface. More generally, let us consider a smooth 4-manifold N which is homotopy equivalent to an Enriques surface. Such a homotopy Enriques surface is known to be homeomorphic to the standard Enriques surface [19], and has a double covering $\pi: K \rightarrow N$ such that K is a homotopy K3 surface. For a Spin^c-structure on $K \rightarrow N$, an O(2)-bundle E called the characteristic O(2)-bundle is associated. Let l_K be the \mathbb{Z} -bundle associated to the double covering $K \rightarrow N$, i.e., $l_K = K \times_{\{\pm 1\}} \mathbb{Z}$. The l_K -coefficient Euler class of E in $H^2(X; l_K)$ is denoted by $\tilde{c}_1(E)$.

Theorem 1.8. *There exists a Spin^c-structure c on $\pi: K \rightarrow N$ which satisfies the following:*

- $\pi^*\tilde{c}_1(E) = 0$, where $\pi^*: H^2(N; l_K) \rightarrow H^2(K; \mathbb{Z})$ is the induced homomorphism.
- the \mathbb{Z}_2 -valued Pin⁻(2)-monopole invariant of (N, c) is nontrivial.

Remark 1.9. The virtual dimension of the moduli space of (N, c) is 0.

Remark 1.10. Theorem 1.8 is proved by Theorem 2.17 which relates the Pin⁻(2)-monopole invariants of N with the Seiberg-Witten invariants of the double covering K , together with the non-vanishing result due to J. Morgan and Z. Szabó [14] for homotopy K3 surfaces.

Next we state a connected-sum formula for Pin⁻(2)-monopole invariants. Before that, we note the following remarks. In general, an ordinary Spin^c-structure can be seen as a reduction of an *untwisted* Spin^c-structure defined on a *trivial* double cover $\tilde{X} \rightarrow X$ (§2(i)). Furthermore, Seiberg-Witten (U(1)-monopole) equations on a Spin^c-structure can be identified with Pin⁻(2)-monopole equations on the corresponding untwisted Spin^c-structure (§2(iv)). Often, we will not distinguish an untwisted Spin^c-structure and the Spin^c-structure which is its reduction, and use the same symbol. In the following, we consider the gluing of Pin⁻(2)-monopoles and ordinary Seiberg-Witten U(1)-monopoles.

Let X_1 be a 4-manifold with an ordinary Spin^c-(or untwisted Spin^c-)structure c_1 . To define another 4-manifold X_2 , let us consider a 2-torus T^2 with a nontrivial \mathbb{Z} -bundle l_T . An oriented Riemann surface Σ with positive genus g can be considered as a connected sum of g tori: $\Sigma = T^2 \# \cdots \# T^2$. Let l_Σ be the \mathbb{Z} -bundle over Σ which is given by the

connected sum of l_T : $l_\Sigma = l_T \# \cdots \# l_T$. Let $\Sigma_1 = T^2$ and Σ_2 be a Riemann surface with positive genus, and consider their direct product $\Sigma_1 \times \Sigma_2$ with a \mathbb{Z} -bundle l which is defined as

$$l = \pi_1^* l_{\Sigma_1} \otimes \pi_2^* l_{\Sigma_2},$$

where $\pi_i: \Sigma_1 \times \Sigma_2 \rightarrow \Sigma_i$ are the projections. We also consider $S^2 \times \Sigma$ with the \mathbb{Z} -bundle l which is the pullback of l_Σ .

Remark 1.11. For $(X; l)$, let $b_k^l = b_k(X; l) = \dim H^k(X; l \otimes \mathbb{Q})$. For $(X, l) = (S^2 \times \Sigma_g; l)$, $b_0^l = b_2^l = b_4^l = 0$ and $b_1^l = b_3^l = 2g - 2$. For $(X, l) = (T^2 \times \Sigma_g; l)$, $b_k^l = 0$ for all k .

For any positive integer i , let us consider $\Sigma_1^{(i)} \times \Sigma_2^{(i)}$ with the \mathbb{Z} -bundle $l^{(i)}$ as above such that $\Sigma_1^{(i)} = S^2$ or T^2 and the genus of $\Sigma_2^{(i)}$ is positive. For a fixed positive k , let us consider the connected sum $X_2 = \#_{i=1}^k (\Sigma_1^{(i)} \times \Sigma_2^{(i)})$ with the \mathbb{Z} -bundle $l_{X_2} = l^{(1)} \# \cdots \# l^{(k)}$. If we write the cardinality of $H^2(X_2; l_{X_2})$ as m , there are m distinct isomorphism classes of Spin^c -structures for $\tilde{X}_2 \rightarrow X_2$, where \tilde{X}_2 is the double covering associated to l_{X_2} . (See Proposition 2.2.) Each of these Spin^c -structures on X_2 has a characteristic $\text{O}(2)$ -bundle E with $\tilde{c}_1(E) = 0$, and therefore E is isomorphic to $\underline{\mathbb{R}} \oplus (l_{X_2} \otimes \mathbb{R})$. Let c_2 be such a Spin^c -structure on X_2 . We consider the connected sum $X_1 \# X_2$ with the Spin^c -structure $c_1 \# c_2$ which is the connected sum of the Spin^c -structures c_1 and c_2 . (Here we assume c_1 as an untwisted Spin^c -structure.) Then, the following holds:

Theorem 1.12. *Let X_1 be a closed oriented connected 4-manifolds with a Spin^c (untwisted Spin^c)-structure such that*

- $b_1(X_1) = 0$, $b_+(X_1) \geq 2$,
- the virtual dimension of the Seiberg-Witten moduli for (X_1, c_1) is zero,
- the Seiberg-Witten invariant for (X_1, c_1) is odd.

Let $X_2 = \#_{i=1}^k (\Sigma_1^{(i)} \times \Sigma_2^{(i)})$ and l_{X_2} be as above. Then, for any Spin^c -structure c_2 on $\tilde{X}_2 \rightarrow X_2$, the $\text{Pin}^-(2)$ -monopole invariant of $(X_1 \# X_2, c_1 \# c_2)$ is nonzero.

Remark 1.13. The virtual dimension d of the moduli space of $(X_1 \# X_2, c_1 \# c_2)$ is positive: If we set $g^{(i)} = g(\Sigma_2^{(i)})$, then $d = 2 \sum_{i=1}^k g^{(i)} - k \geq k$.

Remark 1.14. This non-vanishing result would be interesting in the following two points: First, although the dimension of the moduli is positive, the (co)homological (not cohomotopical) invariant is nontrivial. Second, if $\Sigma_1^{(i)} = S^2$ for some i , all of the Seiberg-Witten invariants and the Seiberg-Witten cohomotopy invariants [2] of $X_1 \# X_2$ are 0 because X_2 admits a positive scalar curvature metric and $b_+(X_2) > 0$.

Remark 1.15. It is worth to notice that $b_+(X_2; l) = 0$. In fact, Theorem 1.12 can be considered as a $\text{Pin}^-(2)$ -monopole analogue of the Seiberg-Witten gluing formulae for connected sums $X_1 \# X_2$ when X_1 is a 4-manifold with $b_+(X_1) > 0$, and

- (1) X_2 is a 4-manifold with $b_1(X_2) = b_+(X_2) = 0$ (e.g., $\overline{\mathbb{CP}}^2$, a rational homology 4-sphere [5, 11]), or
- (2) $X_2 = S^1 \times S^3$, or

(3) X_2 is a connected sum of several manifolds in (1) or (2) above.

As mentioned above, the Pin⁻(2)-monopole invariants are defined as \mathbb{Z}_2 -valued invariants. But in some exceptional cases, we can define \mathbb{Z} -valued invariants. For instance, the non-vanishing result for homotopy Enriques surfaces (Theorem 1.8) is refined as follows:

Theorem 1.16. *The \mathbb{Z} -valued Pin⁻(2)-monopole invariant for (N, c) in Theorem 1.8 is odd.*

Furthermore, the following holds for connected sums of homotopy Enriques surfaces.

Theorem 1.17. *For any integer $n \geq 2$, let $X_n = N_1 \# N_2 \# \cdots \# N_n$ where each N_i is a homotopy Enriques surface. Then X_n has a Spin^c-structure c_n such that*

- \mathbb{Z}_2 -valued Pin⁻(2)-monopole invariant is 0, but
- \mathbb{Z} -valued invariant is nontrivial.

Remark 1.18. Since $b_+(N_i) \geq 1$, Seiberg-Witten invariants and Donaldson invariants of X_n are 0.

Now we state the following general form of the adjunction inequality, which, together with nonvanishing results above, implies Theorem 1.3.

Theorem 1.19. *Let c be a Spin^c-structure on $\tilde{X} \rightarrow X$, and \tilde{c} be the Spin^c-structure on \tilde{X} induced from c (see §2). Suppose at least one of the following occurs:*

- $b_+(X; l) \geq 2$ and the Pin⁻(2)-monopole invariant of (X, c) is nontrivial.
- $b_+(\tilde{X}) \geq 2$ and the ordinary Seiberg-Witten invariant of (\tilde{X}, \tilde{c}) is nontrivial.

Suppose a class $\alpha \in H_2(X; l)$ is represented by a connected embedded surface as above. If α has an infinite order and $\alpha \cdot \alpha \geq 0$, then

$$-\chi(\Sigma) \geq |\tilde{c}_1(E) \cdot \alpha| + \alpha \cdot \alpha,$$

where $\chi(\Sigma)$ is the Euler number of Σ .

The organization of the paper is as follows. In Section 2, we introduce Pin⁻(2)-monopole invariants, and discuss the relation with the Seiberg-Witten invariants on the double covering, and prove Theorem 1.8 and Theorem 1.16. Then several versions of gluing formulae are stated, and assuming these, we prove Theorem 1.17 and Theorem 1.1. Sections 3-5 are devoted to the proof of the gluing theorems stated in §2. Section 3 describes the Pin⁻(2)-monopole theory on 3-manifolds. Section 4 deals with finite energy Pin⁻(2)-monopoles on 4-manifolds with tubular ends. In Section 5, we give proofs of the gluing theorems. In Section 6, the proof of the genus estimate (Theorem 1.19) is given, and Section 7 is a discussion on the genus estimate. The Appendix provides some analytic detail of the gluing construction.

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2. $\text{Pin}^-(2)$ -monopole invariants

2(i). **Spin^{c-} -structures.** As mentioned in the introduction, the $\text{Pin}^-(2)$ -monopole equations are defined on Spin^{c-} -structures, which are a $\text{Pin}^-(2)$ -version of the Spin^c -structures. While a Spin^c -structure is given as a $\text{Spin}^c(4) = \text{Spin}(4) \times_{\{\pm 1\}} \text{U}(1)$ -lift of the frame bundle, a Spin^{c-} -structure is given by a $\text{Spin}(4) \times_{\{\pm 1\}} \text{Pin}^-(2)$ -lift of the frame bundle. The group $\text{Spin}(4) \times_{\{\pm 1\}} \text{Pin}^-(2)$ is denoted by $\text{Spin}^{c-}(4)$. The precise definition is given as follows. (See also [17], Section 3.) Let X be a closed oriented connected Riemannian 4-manifold with double covering $\tilde{X} \rightarrow X$. The $\text{SO}(4)$ -frame bundle on X is denoted by $Fr(X)$. Since $\text{Pin}^-(2) = \text{U}(1) \cup j\text{U}(1)$, $\text{Spin}^c(4)$ is the identity component of $\text{Spin}^{c-}(4)$, and $\text{Spin}^{c-}(4)/\text{Spin}^c(4) = \{\pm 1\}$. Also we have $\text{Spin}^{c-}(4)/\text{Pin}^-(2) = \text{SO}(4)$ and $\text{Spin}^{c-}(4)/\text{Spin}(4) = \text{O}(2)$.

Definition 2.1. A Spin^{c-} -structure on $\tilde{X} \rightarrow X$ is a triple (P, σ, τ) where

- P is a $\text{Spin}^{c-}(4)$ -bundle over X ,
- σ is an isomorphism between $\mathbb{Z}/2$ -bundles $P/\text{Spin}^c(4)$ and \tilde{X} ,
- τ is an isomorphism between $\text{SO}(4)$ -bundles $P/\text{Pin}^-(2)$ and $Fr(X)$.

Instead of the determinant $\text{U}(1)$ -bundle for a Spin^c -structure, an $\text{O}(2)$ -bundle $E = P/\text{Spin}^c(4)$ is associated to a Spin^{c-} -structure. We call this E the *characteristic* $\text{O}(2)$ -bundle. Let l be the \mathbb{Z} -bundle $\tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$ over X . Then l is related to E by $\det E = l \otimes \mathbb{R}$. The basic fact on Spin^{c-} -structure on $\tilde{X} \rightarrow X$ is as follows:

Proposition 2.2. (1) For an $\text{O}(2)$ -bundle E over X with $\det E = l \otimes \mathbb{R}$ as above, there exists a Spin^{c-} -structure on $\tilde{X} \rightarrow X$ whose characteristic bundle is isomorphic to E if and only if $w_2(X) = w_2(E) + w_1(l \otimes \mathbb{R})^2$.

(2) If a Spin^{c-} -structure on $\tilde{X} \rightarrow X$ is given, there is a bijective correspondence between the set of isomorphism classes of Spin^{c-} -structures on $\tilde{X} \rightarrow X$ and $H^2(X; l)$.

Proof. The assertion (1) is proved in [17]. To prove the assertion (2), let us consider the exact sequence,

$$(2.3) \quad 1 \rightarrow S^1 \rightarrow \text{Spin}^{c-}(4) \rightarrow \text{SO}(4) \times \{\pm 1\} \rightarrow 1.$$

From this, we have a fibration,

$$(2.4) \quad BS^1 \rightarrow B\text{Spin}^{c-}(4) \rightarrow B(\text{SO}(4) \times \{\pm 1\}).$$

In (2.3), $\{\pm 1\}$ gives rise to an automorphism of S^1 of complex conjugation. If we identify BS^1 with \mathbb{CP}^∞ , the action of $\pi_1(B(\{\pm 1\})) \cong \mathbb{Z}_2$ on a fiber of (2.4) can be homotopically identified with complex conjugation on \mathbb{CP}^∞ . Then Spin^{c-} -structures on $\tilde{X} \rightarrow X$ are classified by

$$H^2(X; \tilde{\pi}_2(BS^1)) \cong H^2(X; l),$$

where $\tilde{\pi}_2$ is the local coefficient with respect to the $\pi_1(B(\{\pm 1\}))$ -action on fibers. \square

Usually, we will assume the covering $\tilde{X} \rightarrow X$ is nontrivial. But in the case when $\tilde{X} \rightarrow X$ is trivial, the $\text{Spin}^{c-}(4)$ -bundle of a Spin^{c-} -structure on X has a $\text{Spin}^c(4)$ -reduction, and in

fact, this reduction induces a Spin^c -structure on X . We will refer to a Spin^c -structure with trivial \tilde{X} as an *untwisted* Spin^c -structure. We will often make no distinction between an untwisted Spin^c -structure and the Spin^c -structure obtained by reduction. Furthermore, we will see later that the $\text{Pin}^-(2)$ -monopole solutions on an untwisted Spin^c -structure are also reduced to the Seiberg-Witten solutions on the Spin^c -structure which is given by reduction (Proposition 2.14). A typical example of untwisted Spin^c -structure appears when we pullback a (twisted) Spin^c -structure on X to the double covering \tilde{X} . In such a situation, we can relate the $\text{Pin}^-(2)$ -monopole theory on X with the Seiberg-Witten theory on the double covering \tilde{X} with a certain antilinear involution I . (See §2(v).)

2(ii). **Definition of $\text{Pin}^-(2)$ -monopole invariants.** In this subsection, we introduce $\text{Pin}^-(2)$ -monopole invariants. Let X be an oriented closed connected 4-manifold with double covering $\tilde{X} \rightarrow X$, and suppose a Spin^c -structure c on $\tilde{X} \rightarrow X$ is given. Let $l = \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$, $\lambda = l \otimes \mathbb{R}$, and E be the characteristic $\text{O}(2)$ -bundle. Then we have $\lambda = \det E$. In order to define $\text{Pin}^-(2)$ -monopole invariants, we need in general to perturb the equations. As in the Seiberg-Witten case, a way of perturbation is to add a twisted self-dual 2-form $\mu \in \Omega^+(i\lambda)$ as follows.

$$(2.5) \quad \begin{cases} D_A \Phi = 0, \\ \frac{1}{2} F_A^+ = q(\Phi) + \mu. \end{cases}$$

Here we adopt the convention according to [13], slightly different from [17], with $\frac{1}{2}$ on the curvature F_A^+ . The gauge transformation group is given by $\mathcal{G} = \Gamma(\tilde{X} \times_{\{\pm 1\}} \text{U}(1))$, where $\{\pm 1\}$ acts on $\text{U}(1)$ by complex conjugation. The moduli space $\mathcal{M}(X, c) = \mathcal{M}_{\text{Pin}^-(2)}(X, c)$ is defined as the space of solutions modulo gauge transformations.

Remark 2.6. When the Spin^c -structure is untwisted, since $\tilde{X} \rightarrow X$ is trivial, we have $\mathcal{G} = \Gamma(\tilde{X} \times_{\{\pm 1\}} \text{U}(1)) \cong \text{Map}(X, \text{U}(1))$. While the stabilizer of the $\text{Pin}^-(2)$ -monopole reducible on a twisted Spin^c -structure is $\{\pm 1\}$, that in the untwisted case is $\text{U}(1)$. (See also §2(iv).)

For the time being, we suppose the Spin^c -structure is twisted. Suppose $b_+(X; l) \geq 1$. Then, as in the case of the ordinary Seiberg-Witten theory, by a generic choice of μ , the moduli space $\mathcal{M}(X, c)$ have no reducible and is a compact manifold whose dimension is given by

$$(2.7) \quad d(c) = \frac{1}{4}(\tilde{c}_1(E)^2 - \text{sign}(X)) - (b_0(X; l) - b_1(X; l) + b_+(X; l)).$$

(The perturbed moduli space will be denoted by the same symbol $\mathcal{M}(X, c)$.) Let \mathcal{A} be the space of $\text{O}(2)$ -connection on E , \mathcal{C} the configuration space $\mathcal{C} = \mathcal{A} \times \Gamma(S^+)$, and \mathcal{C}^* the space of irreducible configurations $\mathcal{C}^* = \mathcal{A} \times (\Gamma(S^+) \setminus 0)$. Let $\mathcal{B}^* = \mathcal{C}^* / \mathcal{G}$. Then $\mathcal{M}(X, c)$ is embedded in \mathcal{B}^* . In a sense, the $\text{Pin}^-(2)$ -monopole invariant of (X, c) is defined as the fundamental class of the moduli space $[\mathcal{M}(X, c)] \in H_{d(c)}(\mathcal{B}^*)$. We can obtain a numerical invariant by evaluating $[\mathcal{M}(X, c)]$ by a cohomology class in $H^{d(c)}(\mathcal{B}^*)$. If $\tilde{X} \rightarrow X$ is nontrivial, \mathcal{B}^* has

the homotopy type of the classifying space of $\mathbb{Z}/2 \times \mathbb{Z}^{b_1(X;l)}$ ([17], Proposition 4.20). In contrast to the ordinary Seiberg-Witten theory, the moduli space $\mathcal{M}(X, c)$ may be non-orientable. (A necessary condition for $\mathcal{M}(X, c)$ to be orientable will be given in §2(iii).)

In general, we can define the following $\mathbb{Z}/2$ -valued version of the $\text{Pin}^-(2)$ -monopole invariants.

Definition 2.8. The $\text{Pin}^-(2)$ -monopole invariant of (X, c) is defined as a map

$$\text{SW}^{\text{Pin}}(X, c): H^{d(c)}(\mathcal{B}^*; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2,$$

given by

$$\text{SW}^{\text{Pin}}(X, c)(\xi) := \langle \xi, [\mathcal{M}(X, c)] \rangle.$$

If $b_+(X; l) \geq 2$, then $\text{SW}^{\text{Pin}}(X, c)$ is a diffeomorphism invariant. If $b_+(X; l) = 1$, then $\text{SW}^{\text{Pin}}(X, c)$ depends on the chamber structure of the space of metrics and perturbations.

Remark 2.9. The compactness of $\mathcal{M}(X, c)$ enables us to develop the Bauer-Furuta theory [2] for the $\text{Pin}^-(2)$ -monopole equations. In fact, we can define a stable cohomotopy refinement of the $\text{Pin}^-(2)$ -monopole invariants. This will be discussed elsewhere.

2(iii). Orientability of the moduli spaces. The purpose of this subsection is to discuss the orientability of the moduli spaces. Let us consider the family of Dirac operators $\tilde{\delta}_{\text{Dirac}} = \{D_A\}_{A \in \mathcal{A}}$. In [17], §4, we introduced a subgroup \mathcal{K}_γ in \mathcal{G} , which has the properties:

- $\mathcal{G}/\mathcal{K}_\gamma = \{\pm 1\}$.
- \mathcal{K}_γ acts on \mathcal{A} freely, and $\mathcal{A}/\mathcal{K}_\gamma$ can be identified with $H^1(X; \lambda)/H^1(X; l)$.

Remark 2.10. Here γ is a circle embedded in X on which l is nontrivial. The subgroup \mathcal{K}_γ is the set of gauge transformations whose restrictions to γ are homotopic to 1.

Dividing $\tilde{\delta}_{\text{Dirac}}$ by \mathcal{K}_γ , we obtain the family $\delta_{\text{Dirac}} = \tilde{\delta}_{\text{Dirac}}/\mathcal{K}_\gamma$ over $\mathcal{A}/\mathcal{K}_\gamma$.

Proposition 2.11. *If the index of the Dirac operator is even and $\det \text{ind } \delta_{\text{Dirac}}$ is trivial, then the moduli space is orientable.*

Proof. For a configuration (A, Φ) , let us consider the sequence,

$$0 \longrightarrow \Omega^0(i\lambda) \xrightarrow{\mathcal{I}_\Phi} \Omega^1(i\lambda) \oplus \Gamma(S^+) \xrightarrow{\mathcal{D}_{(A, \Phi)}} (\Omega^+(i\lambda) \oplus \Gamma(S^-)) \longrightarrow 0,$$

where $\mathcal{I}(f) = (-2df, f\Phi)$ and $\mathcal{D}_{(A, \Phi)}(a, \phi) = d^+a - Dq_\Phi(\phi), D_A\phi + \frac{1}{2}\rho(a)\Phi$, which are the linearizations of the gauge group action and the monopole map. Let $V = \Omega^1(i\lambda) \oplus \Gamma(S^+)$, and $W = (\Omega^0 \oplus \Omega^+)(i\lambda) \oplus \Gamma(S^-)$ and define $\delta_{(A, \Phi)}: V \rightarrow W$ by,

$$\delta_{(A, \Phi)} = \mathcal{I}_\Phi^* \oplus \mathcal{D}_{(A, \Phi)}.$$

Then the family $\tilde{\delta} = \{\delta_{(A, \Phi)}\}_{(A, \Phi) \in \mathcal{C}}$ defines a bundle homomorphism between the bundles over \mathcal{C} ,

$$\tilde{\delta}: \mathcal{C} \times V \rightarrow \mathcal{C} \times W.$$

Restricting $\tilde{\delta}$ to \mathcal{C}^* and dividing by \mathcal{G} , we obtain a bundle homomorphism over $\mathcal{B}^* = \mathcal{C}^*/\mathcal{G}$,

$$\delta: \mathcal{C}^* \times_{\mathcal{G}} V \rightarrow \mathcal{C}^* \times_{\mathcal{G}} W.$$

The moduli space is orientable if $\det \text{ind } \delta$ is trivial. By deforming $\delta_{(A,\Phi)}$ by $\delta_{(A,t\Phi)}$ ($0 \leq t \leq 1$), we may assume $\tilde{\delta} = \{(d^* \oplus d^+) \oplus D_A\}_{(A,\Phi) \in \mathcal{C}}$. Since $(d^* \oplus d^+)$ does not depend on (A, Φ) , $\det \text{ind}(d^* \oplus d^+)$ is trivial. Therefore it suffices to consider the Dirac family

$$(2.12) \quad \tilde{\delta}' = \{D_A\}_{(A,\Phi) \in \mathcal{C}}: \mathcal{C} \times \Gamma(S^+) \rightarrow \mathcal{C} \times \Gamma(S^-).$$

Then (2.12) can be identified with the pull-back of $\tilde{\delta}_{Dirac}$, via the projection $p: \mathcal{C} \rightarrow \mathcal{A}$ with $p(A, \Phi) = A$. Dividing (2.12) by \mathcal{K}_γ , we obtain $\tilde{\delta}'/\mathcal{K}: \mathcal{C} \times_{\mathcal{K}_\gamma} \Gamma(S^+) \rightarrow \mathcal{C} \times_{\mathcal{K}_\gamma} \Gamma(S^-)$. Note that $\mathcal{C}/\mathcal{K}_\gamma$ is homotopic to $\mathcal{A}/\mathcal{K}_\gamma$. Thus $\text{ind}(\tilde{\delta}'/\mathcal{K})$ is identified with $p^* \text{ind}(\delta_{Dirac})$, which is trivial by the assumption. Hence $\det \text{ind } \delta$ is trivial if and only if $\det((p^* \text{ind}(\delta_{Dirac}))|_{\mathcal{C}^*})/\{\pm 1\}$ over $\mathcal{C}^*/\mathcal{G}$ is trivial. Note that $\mathcal{C}^*/\mathcal{G} \simeq \mathbb{RP}^\infty \times T^{b_1(X;l)}$. Let $\eta \rightarrow \mathcal{C}^*/\mathcal{G}$ be the nontrivial real line bundle which represents the generator of $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$. Then by the assumptions, we see that $\det((p^* \text{ind}(\delta_{Dirac}))|_{\mathcal{C}^*})/\{\pm 1\} \cong \eta^{\otimes \text{ind } D}$. Thus the proposition is proved. \square

Remark 2.13. For instance, if $b_1(X;l) = 0$ and the Dirac index is even, then the moduli space is orientable.

Note that $H^*(\mathcal{B}^*; \mathbb{Z})/\text{Tor} \cong H^*(T^{b_1(X;l)}; \mathbb{Z})$. Suppose the moduli space $\mathcal{M}(X)$ is orientable. Fixing an orientation, we can define \mathbb{Z} -valued Pin⁻(2)-monopole invariants $\text{SW}_{\mathbb{Z}}^{\text{Pin}^-}$ by evaluating the fundamental class $[\mathcal{M}(X)]$ by infinite-order classes ξ in $H^*(\mathcal{B}^*; \mathbb{Z})$:

$$\text{SW}_{\mathbb{Z}}^{\text{Pin}^-}(X, c)(\xi) = \langle \xi, [\mathcal{M}(X)] \rangle.$$

2(iv). **Pin⁻(2)-monopoles on untwisted Spin^c-structures.** Let us consider an untwisted Spin^c-structure $c = (P, \sigma, \tau)$, i.e., a Spin^c-structure on a trivial double covering $\tilde{X} \rightarrow X$. Since $P/\text{Spin}^c(4) \cong \tilde{X} \cong X \times \{\pm 1\}$, there are two Spin^c(4)-reductions which induce two Spin^c-structures, c' and c'' . Then these Spin^c-structures c' and c'' are mutually complex conjugate. (These two may be isomorphic.) In fact, the projection $P \rightarrow P/\text{Spin}^c(4) \cong X \times \{\pm 1\}$ defines a Spin^c-structure on $X \times \{\pm 1\}$, and its restrictions to the connected components are mutually complex conjugate Spin^c-structures (see [17], §2(iii)). As *real* vector bundles, we have identifications among spinor bundles for c , c' and c'' ,

$$S_c^\pm \cong S_{c'}^\pm \cong S_{c''}^\pm.$$

Also as *real* vector bundles, we have identifications among the \mathbb{R}^2 -vector bundle associated to the characteristic O(2)-bundle E of c and the determinant line bundles $L_{c'}$ and $L_{c''}$. If an O(2)-connection A on E is given, we have U(1)-connections A' and A'' on $L_{c'}$ and $L_{c''}$ induced from A by reduction. As *real* operators, covariant derivatives of A , A' and A'' can be identified, and therefore the Dirac operators induced from A , A' and A'' can also be identified as real operators. Furthermore, it can be seen that Pin⁻(2)-monopole solutions on c can be identified with Seiberg-Witten solutions on c' and c'' via the identifications above:

Proposition 2.14. *Let c be an untwisted Spin^c-structure, and c' and c'' the Spin^c-structures which are its reductions. Then there are identifications among the set of Pin⁻(2)-monopole solutions on c and the sets of Seiberg-Witten solutions on c' and c'' . Moreover,*

at the level of moduli spaces, we have

$$\mathcal{M}_{\text{Pin}^-(2)}(X, c) \cong \mathcal{M}_{\text{U}(1)}(X, c') \cong \mathcal{M}_{\text{U}(1)}(X, c''),$$

where $\mathcal{M}_{\text{U}(1)}$ means the ordinary Seiberg-Witten (U(1)-monopole) moduli spaces.

2(v). Relation with the Seiberg-Witten invariants of the double coverings. Let us consider a Spin^{c-} -structure c on a *nontrivial* covering $\pi: \tilde{X} \rightarrow X$. If we pull-back the Spin^{c-} -structure c to \tilde{X} , the pulled-back Spin^{c-} -structure \tilde{c} on \tilde{X} is *untwisted*. If P is the $\text{Spin}^{c-}(4)$ -bundle for c , the projection $P \rightarrow P/\text{Spin}^c \cong \tilde{X}$ can be considered as a $\text{Spin}^c(4)$ -bundle over \tilde{X} which defines a Spin^c -structure \tilde{c}' over \tilde{X} which is obtained by reduction from \tilde{c} . Then π^*P is identified with $P \times_{\text{Spin}^c(4)} \text{Spin}^{c-}(4)$. The covering transformation $\iota: \tilde{X} \rightarrow \tilde{X}$ has a natural lift $\tilde{\iota}$ on \tilde{c} which is given by a $\text{Spin}^{c-}(4)$ -bundle morphism of $P \times_{\text{Spin}^c(4)} \text{Spin}^{c-}(4)$ defined by $\tilde{\iota}([p, g]) = [pJ, J^{-1}g]$ for $[p, g] \in P \times_{\text{Spin}^c(4)} \text{Spin}^{c-}(4)$, where $J = [1, j^{-1}] \in \text{Spin}^{c-}(4) = \text{Spin}(4) \times_{\{\pm 1\}} \text{Pin}^-(2)$. Then there is a bijective correspondence between the configuration space of c and the space of $\tilde{\iota}$ -invariant configurations on \tilde{c} . If we interpret the objects on \tilde{c} in terms of the Spin^c -structure \tilde{c}' of reduction, the $\tilde{\iota}$ -action is identified with the antilinear involution I defined in [17], §4(v). Thus we can identify configurations on (X, c) with I -invariant configurations on (\tilde{X}, \tilde{c}') . In particular, we have,

Proposition 2.15 ([17], Proposition 4.11). *There is a bijective correspondence between the set of $\text{Pin}^-(2)$ -monopole solutions on (X, c) and the set of I -invariant Seiberg-Witten solutions on (\tilde{X}, \tilde{c}') . Moreover we have*

$$(2.16) \quad \mathcal{M}_{\text{Pin}^-(2)}(X, c) \cong \mathcal{M}_{\text{U}(1)}(\tilde{X}, \tilde{c}')^I.$$

Let us discuss the relation of the $\text{Pin}^-(2)$ -monopole invariants of X and the Seiberg-Witten invariants of \tilde{X} . Mimicking the arguments in [21] or [16], we can prove a formula which relates the $\text{Pin}^-(2)$ -monopole invariants of (X, c) with the Seiberg-Witten invariants of (\tilde{X}, \tilde{c}') as follows.

Theorem 2.17. *If $d(c) = 0$ and $b_1(\tilde{X}) = 0$, then*

$$(2.18) \quad \text{SW}^{\text{U}(1)}(\tilde{X}, \tilde{c}') \equiv \sum_{c_\sigma} \text{SW}^{\text{Pin}}(X, c_\sigma) \pmod{2}$$

where $\text{SW}^{\text{U}(1)}(\tilde{X}, \tilde{c}')$ is the Seiberg-Witten invariant of (\tilde{X}, \tilde{c}') , and c_σ runs through all Spin^{c-} -structures on X whose pull-back on \tilde{X} are isomorphic to \tilde{c} , the pull-back of c . Furthermore, if the $\text{Pin}^-(2)$ -monopole moduli spaces are orientable, then the \mathbb{Z} -valued $\text{Pin}^-(2)$ -monopole invariants $\text{SW}_{\mathbb{Z}}^{\text{Pin}}$ also satisfies the relation (2.18).

Remark 2.19. Since the I -action is free and $d(c) = 0$, the virtual dimension of the Seiberg-Witten moduli for (\tilde{X}, \tilde{c}') is also zero.

Remark 2.20. The set of c_σ 's as above is identified with

$$\{c + a \mid a \in \ker(\pi^*: H^*(X; l) \rightarrow H^*(\tilde{X}; \pi^*l))\}.$$

Proof of Theorem 2.17. In the I -equivariant setting, the moduli space $\mathcal{M}_{U(1)}(\tilde{X}, \tilde{c})$ is decomposed into the I -invariant part and the free part. The I -invariant part is identified with $\mathcal{M}_{\text{Pin}^-(2)}(X, c)$ as in (2.16). On the other hand, if the free part is a 0-dimensional manifold, then the number of elements in the free part is even, because $\mathbb{Z}/2$ acts freely. Now, the theorem follows if the equivariant transversality can be achieved by an equivariant perturbation. This issue is discussed in [16]. (Cf. [21].) It is easy to achieve the transversality on the free part. For the I -invariant part, on each point $\xi \in \mathcal{M}_{U(1)}(\tilde{X}, \tilde{c})^I$, let us consider the Kuranishi model $f_\xi: H_1 \rightarrow H_2$, where H_1 and H_2 are finite dimensional I -linear vector spaces. Since the I -action on the base space \tilde{X} is free, the Lefschetz formula tells us that H_1 and H_2 are isomorphic as the I -spaces. Then fixing an I -linear isomorphism $L_\xi: H_1 \rightarrow H_2$, we can perturb the equations I -equivariantly by using L to achieve the transversality around ξ . \square

Now, we can prove Theorem 1.8 and Theorem 1.16.

Proof of Theorem 1.8 and Theorem 1.16. There exists a Spin^c -structure c on N whose associated $O(2)$ -bundle is isomorphic to $\underline{\mathbb{R}} \oplus (l_K \otimes \mathbb{R})$. Then the associated Spin^c -structure \tilde{c} on the double cover K has a trivial determinant line bundle. Then $\text{SW}^{U(1)}(K, \tilde{c})$ is congruent to one modulo 2 by Morgan-Szabó [14]. On the other hand, since $b_1(N; l) = 0$, the Dirac index is even and $d(c) = 0$ for the Spin^c -structure c , the moduli space is orientable, and by fixing an orientation, \mathbb{Z} -valued invariant is defined. Then, by Theorem 2.17, there is a Spin^c -structure c' such that $\text{SW}_{\mathbb{Z}}^{\text{Pin}}(N, c')$ is odd. \square

Remark 2.21. At present, the author does not know the exact value of $\text{SW}_{\mathbb{Z}}^{\text{Pin}}(N, c')$ for any homotopy Enriques surface N .

2(vi). **Gluing formulae.** In this subsection, we state several gluing formulae for $\text{Pin}^-(2)$ -monopole invariants, which will be proved in later sections. The formulae have different forms whether the Spin^c -structures are twisted or untwisted, and the moduli spaces contain reducibles or not. Since \mathcal{B}^* is homotopy equivalent to $\mathbb{RP}^\infty \times T^{b_1(X; l)}$ for a twisted Spin^c -structure, we have an identification

$$H^*(\mathcal{B}^*; \mathbb{Z}_2) = H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \otimes H^*(T^{b_1(X; l)}; \mathbb{Z}_2).$$

Let η be the generator of $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$. For local coefficients l_1 and l_2 over X_1 and X_2 , if both of l_i are nontrivial, then we have $b_1(X_1 \# X_2; l_1 \# l_2) = b_1(X_1; l_1) + b_1(X_2; l_2) + 1$ by the Meyer-Vietris sequence. Hence, for $X = X_1 \# X_2$ and $l = l_1 \# l_2$, $T^{b_1(X; l)}$ is written as

$$(2.22) \quad T^{b_1(X; l)} = T^{b_1(X_1; l_1)} \times T^{b_1(X_2; l_2)} \times T_0,$$

where T_0 is a circle. On the other hand, if one of l_i is trivial, then $T^{b_1(X; l)} = T^{b_1(X_1; l_1)} \times T^{b_1(X_2; l_2)}$. Let t_i^{top} be a generator of $H^{b_1(X_i; l_i)}(T^{b_1(X_i; l_i)})$ for each $i = 1, 2$, and t_0 be a generator of $H^1(T_0)$.

The first gluing formula is on the gluing of $U(1)$ -irreducibles monopoles and $\text{Pin}^-(2)$ -monopoles.

Theorem 2.23. *Let X_1 be a closed oriented connected 4-manifold with a Spin^c (untwisted Spin^{c-})-structure c_1 which satisfies the following:*

- $b_+(X_1) \geq 2$,
- the virtual dimension of $\mathcal{M}_{U(1)}(X_1, c_1)$ is zero,
- $\text{SW}^{U(1)}(X_1, c_1)$ is odd.

Let X_2 be a closed oriented connected 4-manifold which satisfies the following:

- There exists a nontrivial double covering $\tilde{X}_2 \rightarrow X_2$ with $b_+(X_2; l_2) = 0$ where $l_2 = \tilde{X}_2 \times_{\{\pm 1\}} \mathbb{Z}$.
- There exists a Spin^{c-} -structure c_2 on $\tilde{X}_2 \rightarrow X_2$ such that $\tilde{c}_1(E)^2 = 0$ and the Dirac index is zero.

Then, the $\text{Pin}^-(2)$ -monopole invariant for $(X_1 \# X_2, c_1 \# c_2)$ is nontrivial as

$$\text{SW}^{\text{Pin}}(\eta \otimes t_2^{\text{top}}) \neq 0.$$

Theorem 1.12 is a corollary of Theorem 2.23.

The second one is on the gluing of $\text{Pin}^-(2)$ -irreducibles and $U(1)$ -reducibles, that is, a blow-up formula.

Theorem 2.24 (Cf. [5, 18, 7]). *Let X be a closed 4-manifold with a Spin^{c-} -structure c with $b_+(X; l) \geq 2$. Let c' be a Spin^c (untwisted Spin^{c-})-structure on $\overline{\mathbb{CP}}^2$ whose c_1 is a generator of $H^2(\overline{\mathbb{CP}}^2; \mathbb{Z})$. For every $\xi \in H^{d(c)}(\mathcal{B}^*; \mathbb{Z}/2)$,*

$$\text{SW}^{\text{Pin}}(X \# \overline{\mathbb{CP}}^2, c \# c')(\xi) = \text{SW}^{\text{Pin}}(X, c)(\xi).$$

The third one is on the gluing of $\text{Pin}^-(2)$ -irreducibles and $\text{Pin}^-(2)$ -reducibles.

Theorem 2.25. *Let X_1 be a closed oriented connected 4-manifold with a twisted Spin^{c-} -structure c_1 with $b_+(X; l) \geq 2$, and X_2 be a manifold in Theorem 2.23. Let l be the local coefficient associated with $c_1 \# c_2$. Then, for any $\xi \in H^*(\mathcal{B}^*(X_1))$,*

$$\text{SW}^{\text{Pin}}(X_1 \# X_2, c_1 \# c_2)(\xi \otimes t_2^{\text{top}} \otimes t_0) = \text{SW}^{\text{Pin}}(X_1, c_1)(\xi).$$

If 4-manifolds X_1 and X_2 have positive b_+ , then the Seiberg-Witten invariants of $X_1 \# X_2$ are always 0. Likewise, \mathbb{Z}_2 -valued $\text{Pin}^-(2)$ -monopole invariants have similar properties.

Theorem 2.26. *Let X_1 be a closed oriented connected 4-manifold with a twisted Spin^{c-} -structure c_1 with $b_+(X_1; l_1) \geq 1$. Let X_2 be a closed oriented connected 4-manifold with a (twisted/untwisted) Spin^{c-} -structure c_2 , and suppose one of the following:*

- (1) $b_+(X_2) \geq 1$ and c_2 is an untwisted Spin^{c-} -structure on X_2 with $d(c_2) = 0$.
- (2) c_2 is a twisted Spin^{c-} -structure on X_2 with $b_+(X_2; l_2) \geq 1$.

Then $\text{SW}^{\text{Pin}}(X_1 \# X_2, c_1 \# c_2)(\xi) = 0$ for any class $\xi \in H^(\mathcal{B}; \mathbb{Z}_2)$.*

On the other hand, \mathbb{Z} -valued invariants can be nontrivial for a connected sum $X_1 \# X_2$ with positive $b_+(X_1; l_1)$ and $b_+(X_2; l_2)$. This implies Theorem 1.17.

Theorem 2.27. *Let n be any positive integer. For $i = 0, 1, \dots, n$, let X_i be a closed oriented connected 4-manifold with a twisted Spin^{c-} -structure c_i satisfying*

- $b_1(X_i; l_i) = 0$, $b_+(X_i; l_i) \geq 2$.
- $d(c_i) = 0$, and

- the index of the Dirac operator is positive and even.

Note that in this situation, the moduli space $\mathcal{M}(X_i, c_i)$ is orientable, and the \mathbb{Z} -valued invariant $\text{SW}_{\mathbb{Z}}^{\text{Pin}}(X_i, c_i)$ is defined for a choice of orientation. Let $m_i = \text{SW}_{\mathbb{Z}}^{\text{Pin}}(X_i, c_i)$. Then the glued moduli space $\mathcal{M}(X_0 \# \cdots \# X_n, c_1 \# \cdots \# c_n)$ is orientable, and

$$\text{SW}_{\mathbb{Z}}^{\text{Pin}}(X_0 \# \cdots \# X_n, c_1 \# \cdots \# c_n)(t^{\text{top}}) = 2^n \prod_{i=0}^n m_i,$$

for a choice of orientation and a generator $t^{\text{top}} \in H^{b_1}(T^{b_1}; \mathbb{Z})$.

Proof of Theorem 1.17. For each (N_i, l_i) , we have $b_1(X_i; l_i) = 0$ and $b_+(X_i; l_i) = 2$. By Theorem 1.16, there is a twisted Spin^{c-} -structure c_i such that $d(c_i) = 0$, the Dirac index is 2 and $\text{SW}_{\mathbb{Z}}^{\text{Pin}}(X_i, c_i)$ is odd. Then the theorem follows from Theorem 2.27. \square

2(vii). **Proof of Theorem 1.1.** In this subsection, we prove Theorem 1.1 by assuming Theorem 2.23.

Proof of Theorem 1.1. Let (X_2, l_{X_2}) be as in Theorem 1.12. Then this satisfies the conditions for X_2 in Theorem 2.23.

For given n , required exotic structures on $E(n)$ can be constructed by both of logarithmic transformation (see e.g., [9]) and Fintushel-Stern's knot surgery [6].

First, we discuss on the case of logarithmic transformation. Let $E(n)_{p,q}$ be the log transformed $E(n)$ with two multiple fibers of multiplicities p and q . For odd n , all of $E(n)_{p,q}$ with $\gcd(p, q) = 1$ is homeomorphic to $E(n)$. On the other hand, for even n , $E(n)_{p,q}$ is homeomorphic to $E(n)$ if and only if $\gcd(p, q) = 1$ and pq is odd. Let $f \in H^2(E(n)_{p,q})$ be the Poincaré dual of the homology class of a regular fiber. Then there is a primitive class f_0 with $f = pqf_0$, and the Poincaré duals f_p and f_q of the multiple fibers of p and q are given by $f_p = qf_0$ and $f_q = pf_0$. If we put

$$D(a, b, c) = af + bf_p + cf_q,$$

then, for $n \geq 2$, the canonical class K is given as $K = D(n-2, p-1, q-1)$. The Seiberg-Witten basic classes are given by $K - 2D(a, b, c)$, where $0 \leq a \leq n-2$, $0 \leq b \leq p-1$, $0 \leq c \leq q-1$, and the value the Seiberg-Witten invariant for the class $K - 2D(a, b, c)$ is

$$\text{SW}^{\text{U}(1)}(E(n)_{p,q}, K - 2D(a, b, c)) = (-1)^a \binom{n-2}{a},$$

which is independent on b and c . Similar facts hold for the case when $n = 1$. In general, the number of basic classes whose Seiberg-Witten invariants are odd is changed if p and q are varied. By using these facts together with Theorem 2.23, we can find infinitely many $\{p, q\}$ such that $E(n)_{p,q} \# X_2$ have different numbers of basic classes for Pin⁻(2)-monopole invariants.

For a knot K , let $E(n)_K$ be the manifold obtained by the knot surgery on a regular fiber T with K . If we consider the Seiberg-Witten invariant as a symmetric Laurent polynomial as in [6], the invariant of $E(n)$ is related to that of $E(n)_K$ by

$$\text{SW}_{E(n)_K}^{\text{U}(1)} = \text{SW}_{E(n)}^{\text{U}(1)} \cdot \Delta_K(t),$$

where $t = \exp(2[T])$ and $\Delta_K(t)$ is the (symmetrized) Alexander polynomial of K . Now, let $X_K = E(n)_K$, and let us fix a Spin^{c-} -structure c_2 on X_2 as in Theorem 2.23, and consider a function of $\text{Pin}^-(2)$ -monopole invariants of $X_K \# X_2$,

$$\text{SW}_{X_K \# (X_2, c_2)}^{\text{Pin}} : \{h \in H^2(X_K; \mathbb{Z}) \mid h \equiv w_2(X) \pmod{2}\} \rightarrow \mathbb{Z}_2,$$

which is defined as

$$\text{SW}_{X_K \# (X_2, c_2)}^{\text{Pin}}(h) = \text{SW}^{\text{Pin}}(X_K \# X_2, c(h) \# c_2)(\eta \otimes t^{\text{top}}),$$

where $c(h)$ is the Spin^c -structure on X_K with $c_1 = h$. If we assume $\text{SW}_{X_K \# (X_2, c_2)}^{\text{Pin}}$ as a \mathbb{Z}_2 -coefficient polynomial, then Theorem 1.12 implies that $\text{SW}_{X_K \# (X_2, c_2)}^{\text{Pin}}$ is the \mathbb{Z}_2 -reduction of the \mathbb{Z} -coefficient polynomial $\text{SW}_{E(n)_K}^{\text{U}(1)}$. Then we can find infinitely many K so that $\text{SW}_{X_K \# (X_2, c_2)}^{\text{Pin}}$ are different. \square

3. $\text{Pin}^-(2)$ -monopole theory on 3-manifolds

Sections 3-5 are devoted to the proof of the gluing theorems in §2(vi), and this preparatory section is on the $\text{Pin}^-(2)$ -monopole theory on 3-manifolds. We refer to [13, 7] for the Seiberg-Witten counter part of the topics in this section.

3(i). **Spin^{c-} -structures on 3-manifolds.** Let us define the group $\text{Spin}^{c-}(3)$:

$$\text{Spin}^{c-}(3) = \text{Spin}(3) \times_{\{\pm 1\}} \text{Pin}^-(2) = \text{Sp}(1) \times_{\{\pm 1\}} \text{Pin}^-(2).$$

Let Y be an oriented closed connected Riemannian 3-manifold, and $F(Y)$ its $\text{SO}(3)$ -frame bundle. Suppose a double covering $\tilde{Y} \rightarrow Y$ is given. A Spin^{c-} -structure on $\tilde{Y} \rightarrow Y$ is a lift of the $\text{SO}(3)$ -bundle $F(X)$ to a principal $\text{Spin}^{c-}(3)$ -bundle P with an identification that $P/\text{Spin}^c(3) \cong \tilde{Y}$. The characteristic $\text{O}(2)$ -bundle E is the $\text{O}(2)$ -bundle associated to P via the homomorphism $\text{Spin}^{c-}(3) \rightarrow \text{O}(2)$.

Remark 3.1. As in the 4-dimensional case, if $\tilde{Y} \rightarrow Y$ is trivial, then a Spin^{c-} -structure on $\tilde{Y} \rightarrow Y$ can be reduced to a Spin^c -structure on Y , and is called *untwisted*.

Let us define the action of $\text{Spin}^{c-}(3)$ on $\text{Im } \mathbb{H}$ by

$$[q, u] \cdot v = qvq^{-1},$$

for $[q, u] \in \text{Spin}^{c-}(3)$ and $v \in \text{Im } \mathbb{H}$. Then the associated bundle $P \times_{\text{Spin}^{c-}(3)} \text{Im } \mathbb{H}$ is identified with the tangent bundle TY . Let us define the $\text{Spin}^{c-}(3)$ -action on \mathbb{H} by

$$[q, u] \cdot \psi = q\psi u^{-1},$$

for $[q, u] \in \text{Spin}^{c-}(3)$ and $\psi \in \mathbb{H}$. Then we obtain the associated bundle $S = P \times_{\text{Spin}^{c-}(3)} \mathbb{H}$ which is the spinor bundle for the Spin^{c-} -structure.

The Clifford multiplication is defined as follows. The identity component of $\text{Spin}^{c-}(3)$ is a $\text{Spin}^c(3)$, and the quotient group $\text{Spin}^{c-}(3)/\text{Spin}^c(3)$ is isomorphic to $\{\pm 1\}$. Let \mathbb{C}_- be a copy of \mathbb{C} with the $\{\pm 1\}$ -action by complex conjugation. Then $\text{Spin}^{c-}(3)$ acts on \mathbb{C}_- via the projection $\text{Spin}^{c-}(3) \rightarrow \text{Spin}^{c-}(3)/\text{Spin}^c(3) = \{\pm 1\}$. If we define

$$\rho_0 : (\text{Im } \mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}_- \times \mathbb{H} \rightarrow \mathbb{H}$$

by $\rho_0(v \otimes a, \psi) = \bar{v}\psi\bar{a}$, then ρ_0 is $\text{Spin}^{c-}(3)$ -equivariant. Let $K = \tilde{Y} \times_{\{\pm 1\}} \mathbb{C}_-$. Then we can define the Clifford multiplication

$$\rho: T^*Y \otimes_{\mathbb{R}} K \rightarrow \text{Hom}(S, S),$$

which induces

$$\rho: \Omega^1(Y; K) \times \Gamma(S) \rightarrow \Gamma(S).$$

Note that $K = \mathbb{R} \oplus i\lambda$, and so $\Omega^1(Y; K) = \Omega^1(Y; \mathbb{R}) \oplus \Omega^1(Y; i\lambda)$. Although the spinor bundle S does not have an ordinary hermitian inner product, the pointwise *twisted* hermitian product

$$(3.2) \quad \langle \cdot, \cdot \rangle_{K,x}: S_x \times S_x \rightarrow K_x$$

is defined. For $\alpha \otimes 1 \in T^*Y \otimes K$, the image $\rho(\alpha \otimes 1)$ is a traceless endomorphism which is *skew-adjoint* with respect to the inner product (3.2). The whole image of T^*Y by ρ forms the subbundle of $\text{Hom}(S, S)$, which we write as $\mathfrak{su}(S)$, equipped with the inner product $\frac{1}{2} \text{tr}(a^*b)$. When $\{e_1, e_2, e_3\}$ is an oriented frame on $\Lambda^1(Y)$, we assume the orientation convention

$$\rho(e_1)\rho(e_2)\rho(e_3) = 1.$$

We extend ρ to forms by the rule,

$$\rho(\alpha \wedge \beta) = \frac{1}{2}(\rho(\alpha)\rho(\beta) + (-1)^{\deg \alpha \deg \beta} \rho(\beta)\rho(\alpha)).$$

The orientation convention implies $\rho(*\alpha) = -\rho(\alpha)$ for 1-forms.

3(ii). **Pin⁻(2)-monopole equations on 3-manifolds.** Fixing an $O(2)$ -connection B on E and together with the Levi-Civita connection, we obtain a $\text{Spin}^{c-}(3)$ -connection on P , and we can define the Dirac operator $D_B: \Gamma(S) \rightarrow \Gamma(S)$ associated to B .

The bundle $\Lambda^1(Y) \otimes_{\mathbb{R}} i\lambda$ is also associated with P as follows. Let $\varepsilon: \text{Pin}^-(2) \rightarrow \text{Pin}^-(2)/U(1) \cong \{\pm 1\}$ be the projection, and let $\text{Spin}^{c-}(3)$ act on $\text{Im } \mathbb{H}$ by

$$v \in \text{Im } \mathbb{H} \rightarrow \varepsilon(u)qvq^{-1} \quad \text{for } [q, u] \in \text{Spin}^{c-}(3).$$

Then $\Lambda^1(Y) \otimes_{\mathbb{R}} i\lambda$ is identified with $P \times_{\text{Spin}^{c-}(3)} \text{Im } \mathbb{H}$. For $\psi \in \mathbb{H}$, $\psi i \bar{\psi}$ is in $\text{Im } \mathbb{H}$. Then the map $\psi \in \mathbb{H} \rightarrow \psi i \bar{\psi} \in \text{Im } \mathbb{H}$ is $\text{Spin}^{c-}(3)$ -equivariant, and induces a quadratic map

$$q: \Gamma(S) \rightarrow \Omega^1(Y; i\lambda).$$

For a closed 2-form $\eta \in \Omega^2(i\lambda)$, the perturbed Pin⁻(2)-monopole equations on Y are defined as

$$(3.3) \quad \begin{cases} D_B \Psi = 0, \\ -\frac{1}{2}(* (F_B + \eta)) = q(\Psi), \end{cases}$$

for $O(2)$ -connections B on E and $\Psi \in \Gamma(S)$. The gauge transformation group is given by

$$\mathcal{G}_Y = \Gamma(\tilde{Y} \times_{\{\pm 1\}} U(1)),$$

where $\{\pm 1\}$ acts on $U(1)$ by complex conjugation.

Remark 3.4. If the Spin^c -structure is untwisted, then the 3-dimensional $\text{Pin}^-(2)$ -monopole equations are also identified with the 3-dimensional Seiberg-Witten equations.

3(iii). **$\text{Pin}^-(2)$ -Chern-Simons-Dirac functional.** Choose a reference $\text{O}(2)$ -connection B_0 on E . Let $\mathcal{A}(E)$ be the space of $\text{O}(2)$ -connections on E , and $\mathcal{C} = \mathcal{A}(E) \times \Gamma(S)$.

Definition 3.5. Let η be a closed 2-form in $\Omega^2(\lambda)$. The (perturbed) $\text{Pin}^-(2)$ -Chern-Simons-Dirac functional $\vartheta: \mathcal{C} \rightarrow \mathbb{R}$ is defined by

$$(3.6) \quad \vartheta(B, \Psi) = -\frac{1}{8} \int_Y (B - B_0) \wedge (F_B + F_{B_0} + i\eta) + \frac{1}{2} \int_Y \langle D_B \Psi, \Psi \rangle_{\mathbb{R}} \text{dvol}_Y.$$

A few comments on the definition. For $\alpha \in \Omega^1(i\lambda)$ and $\beta \in \Omega^2(i\lambda)$, $\alpha \wedge \beta$ is in $\Omega^3(Y; \mathbb{R})$ since $\lambda^{\otimes 2}$ is trivial. The inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ is the real part of (3.2).

The tangent space of \mathcal{C} at (B, Ψ) is $T_{(B, \Psi)}\mathcal{C} = \Omega^1(i\lambda) \oplus \Gamma(S)$. We equip the tangent space with an L^2 metric. Then the gradient of ϑ with respect to the L^2 -metric is given by

$$\nabla \vartheta = \left(\frac{1}{2}(* (F_B + i\eta)) + q(\Psi), D_B \Psi \right).$$

Hence the critical points of ϑ are the solutions of the $\text{Pin}^-(2)$ -monopole equations on Y .

For $g \in \mathcal{G}_Y$, $g^{-1}dg$ is an $i\lambda$ -valued 1-form, and the λ -valued 1-form $\frac{1}{2\pi i}g^{-1}dg$ represents an integral class $[g] \in H^1(Y; l)/\text{Tor}$.

Proposition 3.7. For $(B, \Psi) \in \mathcal{C}$ and $g \in \mathcal{G}_Y$,

$$\vartheta(g(B, \Psi)) - \vartheta(B, \Psi) = 2\pi([g] \cup (\pi \tilde{c}_1(E) - [\eta])[Y]),$$

where $[\eta] \in H^2(Y; \lambda)$ is the de Rham cohomology class of η .

3(iv). **Nondegenerate critical point on S^3 .** Here, we suppose $Y = S^3$ with a positive scalar curvature metric. Since S^3 is simply-connected, every Spin^c -structure is untwisted. This is unique up to isomorphism and identified with a unique Spin^c -structure. For a positive scalar curvature metric, every monopole solution is a reducible one, say $(\theta, 0)$, which is unique up to gauge. Furthermore, the kernel of the Dirac operator D_θ is trivial. Since the index of D_θ is 0, the cokernel is also trivial, and this implies $(\theta, 0)$ is nondegenerate. The stabilizer of $(\theta, 0)$ of the gauge group action is denoted by Γ_θ :

$$\Gamma_\theta = \{g \in \text{Map}(S^3; \text{U}(1)) \mid g(\theta, 0) = (\theta, 0)\}.$$

Note that $\Gamma_\theta \cong S^1$.

4. $\text{Pin}^-(2)$ -monopoles on a 4-manifold with a tubular end

In this section, we continue the preparation for gluing, and discuss on finite energy $\text{Pin}^-(2)$ -monopoles on 4-manifolds with tubular ends. We refer to [3] as well as [13, 7].

4(i). **Setting.** Let X be a Riemannian 4-manifold with a Spin^{c-} -structure containing a tubular end $[-1, \infty) \times Y$, where Y is a closed, connected, Riemannian 3-manifold with a Spin^{c-} -structure. More precisely, suppose we are given

- (1) an orientation preserving isometric embedding $i: [-1, \infty) \times Y \rightarrow X$ such that

$$X^t = X \setminus i((t, \infty) \times Y)$$

is compact for any $t \geq -1$,

- (2) an isomorphism between Spin^{c-} -structure on $[-1, \infty) \times Y$ induced from Y and the one inherited from X via the embedding i .

Remark 4.1. If the Spin^{c-} -structure on X is twisted but its restriction on the tube $[-1, \infty) \times Y$ is untwisted, then the double cover \tilde{X} has two tubular ends.

Later we will define weighted Sobolev norms on various sections over X . For this purpose, let us take a C^∞ -function $w: X \rightarrow \mathbb{R}$ such that

$$(4.2) \quad w(t) = \begin{cases} 1 & \text{on } X^{-1} \\ e^{\alpha t} & \text{for } (t, y) \in [0, \infty) \times Y \end{cases}$$

where α is a small positive number which will be chosen later to be suitable for our purpose. For $p(\geq 2)$ and a nonnegative integer k , the weighted Sobolev norm of a section f (e.g., a form or a spinor) on X is given by

$$\|f\|_{L_k^{p,w}} = \|wf\|_{L_k^p}.$$

Let X_1 and X_2 be 4-manifolds with tubular ends as above with isometric embeddings

$$i_1: [-1, \infty) \times Y \rightarrow X_1, \quad i_2: [-1, \infty) \times \bar{Y} \rightarrow X_2,$$

where \bar{Y} is Y with opposite orientation. For $T \geq 0$, let $X^{\#T}$ be the manifold obtained by gluing X_1^{2T} and X_2^{2T} via the identification

$$i_1(t, y) \sim i_2(2T - t, y).$$

Then we naturally have an isometric embedding of a neck $i_T: [-T, T] \times Y \rightarrow X^{\#T}$. (Here, the negative side is connected to X_1^0 and the positive side to X_2^0 .) When we take functions w_1, w_2 as (4.2), a continuous function $w_T: X^{\#T} \rightarrow \mathbb{R}$ is induced by gluing w_1 and w_2 such that

$$(4.3) \quad w_T(t) = e^{\alpha(T-|t|)}$$

for $(t, y) \in [-T, T] \times Y$. For the sections over $X^{\#T}$, we will use the weighted norm

$$\|f\|_{L_k^{p,w_T}} = \|w_T f\|_{L_k^p}.$$

4(ii). **Exponential decay.** Since a $\text{Pin}^-(2)$ -monopole on an untwisted Spin^{c-} -structure is identified with an ordinary Seiberg-Witten monopole, the estimates for Seiberg-Witten monopoles on a cylinder $[0, \infty) \times Y$ hold for $\text{Pin}^-(2)$ -monopoles on an untwisted Spin^{c-} -structure. In particular, exponential decay estimates hold. We invoke the results due to Froyshov [7]. (In fact, the following theorems (Theorem 4.4 and Theorem 4.5) can be proved for $\text{Pin}^-(2)$ -monopole on a *twisted* Spin^{c-} -structure.)

Let β be a nondegenerate monopole over Y , and $U \subset \mathcal{B}_Y$ is an L^2 -closed subset which contains no monopoles except perhaps $[\beta]$. Put $K_\beta = \ker \mathcal{I}_\beta$. When H_β is the Hessian at β , let \tilde{H}_β be the restriction of H_β to K_β , $\tilde{H}_\beta = H_\beta|_{K_\beta}: K_\beta \rightarrow K_\beta$. Note that, if β be nondegenerate, then there exist positive numbers λ^\pm such that \tilde{H}_β has no eigenvalue in $[-\lambda^-, \lambda^+]$. Define $B_t = [t-1, t+1] \times Y$.

Theorem 4.4 ([7], Theorem 6.3.1.). *For any $C > 0$, there exist constants ϵ and C_k for nonnegative integer k such that the following holds. Let $x = (A, \Phi)$ be a monopole in temporal gauge over $(-2, \infty) \times Y$ such that $x(t) \in U$ for some $t \geq 0$. Set*

$$\bar{\nu} = \|\nabla \vartheta\|_{L^2((-\infty, \infty) \times Y)}, \quad \nu(t) = \|\nabla \vartheta\|_{L^2(B_t)}.$$

If $\|\Phi\|_\infty \leq C$ and $\bar{\nu} \leq \epsilon$ then there is a smooth monopole α over Y , gauge equivalent to β , such that if B is the connection part of $\pi^ \alpha$ then for every $t \geq 1$ and nonnegative integer k one has*

$$\sup_{y \in Y} |\nabla_B^k (x - \pi^* \alpha)|_{(t, y)} \leq C_k \sqrt{\nu(0)} e^{-\lambda^+ t}.$$

Theorem 4.5 ([7], Theorem 6.3.2.). *For any $C > 0$, there exist constants ϵ and C_k for nonnegative integer k such that the following holds for every $T > 1$. Let $x = (A, \Phi)$ be a monopole in temporal gauge over the band $[-T-2, T+2] \times Y$ such that $x(t) \in U$ for some $t \in [-T-2, T+2]$. Set*

$$\bar{\nu} = \|\nabla \vartheta\|_{L^2([-T-2, T+2] \times Y)}, \quad \nu(t) = \|\nabla \vartheta\|_{L^2(B_t)}.$$

If $\|\Phi\|_\infty \leq C$ and $\bar{\nu} \leq \epsilon$ then there is a smooth monopole α over Y , gauge equivalent to β , such that if B is the connection part of $\pi^ \alpha$ then for every $t \leq T-1$ and nonnegative integer k one has*

$$\sup_{y \in Y} |\nabla_B^k (x - \pi^* \alpha)|_{(t, y)} \leq C_k (\nu(-T) + \nu(T))^{1/2} e^{-\lambda^+(T-|t|)}.$$

Remark 4.6. An easy way to prove similar (but possibly weaker) estimates for $\text{Pin}^-(2)$ -monopoles on twisted Spin^{c-} -structures is lifting everything to the double cover $(0, \infty) \times \tilde{Y}$ on which the corresponding Spin^{c-} -structure is untwisted and applying the estimate for the Seiberg-Witten monopole as above. Of course, we can prove such results for $\text{Pin}^-(2)$ -monopoles by adapting the arguments in [7] *mutatis mutandis*.

4(iii). **Energy.** Let Z be a Riemannian Spin^{c-} -4-manifold possibly noncompact or with boundaries, such as X with a tubular end, or its compact submanifolds X^t or a compact tube $[a, b] \times Y$. Let μ be a closed 2-form in $\Omega^2(i\lambda)$, and assume μ is the pull-back of η on

the tube. For configurations (A, Φ) , we define the energy by

$$\mathcal{E}(A, \Phi) = \frac{1}{4} \int_Z |F_A - \mu|^2 + \int_Z |\nabla_A \Phi|^2 + \frac{1}{4} \int_Z \left(|\Phi|^2 + \frac{s}{2} \right)^2 - \int_Z \frac{s^2}{16} + 2 \int_Z \langle \Phi, \rho(\mu) \Phi \rangle,$$

where s is the scalar curvature.

Proposition 4.7 ([13], Chapter II and Chapter VIII). (1) *If (A, Φ) is a monopole on $Z = X^T$ with a finite cylinder $(-1, T] \times Y$ near the boundary Y , then*

$$\mathcal{E}(A, \Phi) = \frac{1}{4} \int_Z (F_A - \mu) \wedge (F_A - \mu) - \int_Y \langle \Phi|_Y, D_B(\Phi|_Y) \rangle,$$

where B is the boundary connection induced from A .

(2) *If (A, Φ) is a monopole on $[t_0, t_1] \times Y$ in temporal gauge, then*

$$\frac{1}{2} \mathcal{E}(A, \Phi) = \vartheta(A(t_1), \Phi(t_1)) - \vartheta(A(t_0), \Phi(t_0)).$$

4(iv). **Compactness.**

Proposition 4.8 ([13], Theorem 5.1.1). *Let Z be a compact Riemannian Spin^{c-} -4-manifold with boundary. Suppose there exists a constant C so that a sequence (A_n, Φ_n) of smooth solutions to Pin⁻(2)-monopole equations satisfies the bound $\mathcal{E}(A_n, \Phi_n) \leq C$. Then there exists a sequence g_n of (smooth) gauge transformations with the following properties: after passing to a subsequence, the transformed solutions $g_n(A_n, \Phi_n)$ converges weakly in L_1^2 to a L_1^2 -configuration (A, Φ) on Z , and converges strongly in C^∞ on every interior domain $Z' \subset Z$.*

Corollary 4.9. *Let $x(t) = (A(t), \Phi(t))$ be a smooth monopole on $[-1, \infty) \times Y$ in temporal gauge. If $\mathcal{E}(A, \Phi)$ is finite, then $[x(t)]$ converges in \mathcal{B}_Y to some critical point as $t \rightarrow \infty$.*

Proof. By translation, $(A_T, \Phi_T) = (A, \Phi)|_{[T-1, T+1] \times Y}$ can be considered as a monopole on $[-1, 1] \times Y$. Let T_n be any sequence with $T_n \rightarrow \infty$ as $n \rightarrow \infty$. Since $\mathcal{E}(A, \Phi)$ is finite, $\mathcal{E}(A_{T_n}, \Phi_{T_n}) \rightarrow 0$ as $n \rightarrow \infty$. Then, after some gauge transformations, we may assume (A_{T_n}, Φ_{T_n}) converges in C^∞ on $(-1, 1) \times Y$ to the pull-back of some critical point. From this, the corollary is proved. \square

Proposition 4.10. *Let X be a Spin^{c-} -4-manifold X with an end $[-1, \infty) \times Y$. If a smooth monopole (A, Φ) over X has a finite energy $\mathcal{E}(A, \Phi)$, then we have either*

$$\Phi = 0, \quad \text{or} \quad \|\Phi\|_{C^0} \leq -\frac{1}{2} \inf_{x \in X} s(x) + 4\|\mu\|_{C^0},$$

where s is the scalar curvature of X .

Proof. By Corollary 4.9, we may assume (A, Φ) converges to a monopole (B, Ψ) on Y . If $|\Phi|$ takes its maximum on X , then the argument in [12], Lemma 2, implies the proposition. Otherwise we have $\|\Phi\|_{C^0} = \|\Psi\|_{C^0}$. Since (B, Ψ) is a 3-dimensional monopole, Ψ also satisfies

$$\Psi = 0 \quad \text{or} \quad \|\Psi\|_{C^0} \leq -\frac{1}{2} \inf_{y \in Y} s(y) + 4\|\eta\|_{C^0}.$$

\square

4(v). **Weighted Moduli spaces.** Let X be a Spin^{c-} -4-manifold with the end $[-1, \infty) \times S^3$. Let us fix a smooth reference connection A^0 which is the pull-back of θ on the tube $[0, \infty) \times S^3$. For later purpose, we choose p so that

$$(4.11) \quad 2 < p < 4.$$

We consider the space of configurations

$$\mathcal{C}^w = \{(A^0 + a, \Phi) \mid a \in L_k^{p,w}(\Lambda^1(i\lambda)), \Phi \in L_k^{p,w}(S^+)\}.$$

Let us consider the set of gauge transformations

$$\mathcal{G}^w = \{g \in \Gamma(\tilde{X} \times_{\{\pm 1\}} \text{U}(1)) \mid \nabla_0 g \in L_1^{p,w}\},$$

where ∇_0 denotes the covariant derivative of A^0 . We can prove,

Proposition 4.12 ([3], §4.3, Cf. [7] Chapter 2). (1) \mathcal{G}^w is a Banach Lie group which is modeled on the Lie algebra

$$L\mathcal{G}^w = \{\xi \in \Omega^0(i\lambda) \mid \nabla_0 \xi \in L_1^{p,w}\}.$$

with the norm

$$\|\xi\| = \|\nabla_0 \xi\|_{L_1^{p,w}} + |\xi(x_0)|,$$

where $x_0 \in X$ is a fixed base point.

(2) Each element $g \in \mathcal{G}^w$ tends to a limit in Γ_θ at infinity, and therefore the evaluation map is defined:

$$\text{ev}: \mathcal{G}^w \rightarrow \Gamma_\theta.$$

Remark 4.13. The group \mathcal{G}^w is topologized as follows: For the base point x_0 , the fundamental neighborhoods of the identity are given by

$$U_\varepsilon = \{g \in \mathcal{G}^w \mid \|\nabla_0 g\|_{L_1^{p,w}} < \varepsilon, |g(x_0) - 1| < \varepsilon\}.$$

Let \mathcal{G}_0^w be the kernel of ev . Then $\mathcal{G}^w/\mathcal{G}_0^w \cong \Gamma_\theta$. Now the Lie algebra of \mathcal{G}_0^w is given by

$$L\mathcal{G}_0^w = L_2^{p,w}(\Lambda^0(i\lambda)).$$

For a configuration $(A, \Phi) \in \mathcal{C}^w$, the infinitesimal \mathcal{G}_0^w -action is given by the map

$$\mathcal{I}_\Phi: L_2^{p,w}(\Lambda^0(i\lambda)) \rightarrow L_1^{p,w}(\Lambda^1(i\lambda) \oplus S^+)$$

defined by $\mathcal{I}_\Phi(f) = (-df, f\Phi)$. When \mathcal{I}_Φ^* is the formal adjoint of \mathcal{I}_Φ , the adjoint of \mathcal{I}_Φ with respect to the *weighted* norm is given by

$$\mathcal{I}_\Phi^{*,w}(\alpha) = w^{-p} \mathcal{I}_\Phi^*(w^p \alpha).$$

This gives the decomposition (Cf. [7]):

$$L_1^{p,w}(\Lambda^1(i\lambda) \oplus S^+) = (\ker \mathcal{I}_\Phi^{*,w} \subset L_1^{p,w}) \oplus \mathcal{I}_\Phi(L_2^{p,w}).$$

Since the \mathcal{G}_0^w -action on \mathcal{C}^w is free, the quotient space $\tilde{\mathcal{B}}^w = \mathcal{C}^w/\mathcal{G}_0^w$ is a Banach manifold, with a local model

$$T_{[(A,\Phi)]} \tilde{\mathcal{B}}^w = \ker \mathcal{I}_\Phi^{*,w} \cap L_1^{p,w}.$$

The Pin⁻(2)-monopole map is defined as

$$\begin{aligned}\Theta: \mathcal{C}^w &\rightarrow L^{p,w}(\Lambda^+(i\lambda) \oplus S^-), \\ \Theta(A, \Phi) &= (F_A^+ - q(\Phi) - \mu, D_A\Phi).\end{aligned}$$

The moduli space is defined by $\mathcal{M} = \Theta^{-1}(0)/\mathcal{G}^w$.

Proposition 4.14. *The moduli space \mathcal{M} is compact.*

Proof. Let $[(A_n, \Phi_n)]$ be any sequence in \mathcal{M} . In general, one can prove that the sequence has a chain convergent subsequence. ([3], Chapter 5 and [7], Chapter 7.) Since there is only one critical point on $Y = S^3$, the subsequence converges in \mathcal{M} . \square

The differential of Θ at $x = (A, \Phi)$ is given by

$$\begin{aligned}\mathcal{D}_{(A,\Phi)} &= D\Theta: L_1^{p,w}(\Lambda^1(i\lambda) \oplus S^+) \rightarrow L^{p,w}(\Lambda^+(i\lambda) \oplus S^-), \\ \mathcal{D}_{(A,\Phi)}(a, \phi) &= (d^+a - Dq_\Phi(\phi), D_A\phi + \frac{1}{2}\rho(b)\Phi),\end{aligned}$$

where Dq_Φ is the differential of q . Then

$$(4.15) \quad \mathcal{D}_{(A,\Phi)} \circ \mathcal{I}_\Phi(f) = (0, fD_A\Phi).$$

Therefore, if (A, Φ) is a Pin⁻(2)-monopole solution, then $\mathcal{D}_{(A,\Phi)} \circ \mathcal{I}_\Phi(f) = 0$, which forms the deformation complex:

$$0 \longrightarrow L_2^{p,w}(\Lambda^0(i\lambda)) \xrightarrow{\mathcal{I}_\Phi} L_1^{p,w}(\Lambda^1(i\lambda) \oplus S^+) \xrightarrow{\mathcal{D}_{(A,\Phi)}} L^{p,w}(\Lambda^+(i\lambda) \oplus S^-) \longrightarrow 0.$$

The cohomology groups are denoted by $H_{(A,\Phi)}^i$.

The monopole map Θ defines a Γ_θ -invariant section of a bundle over $\tilde{\mathcal{B}}^w$ whose linearization is given by $\mathcal{I}_\Phi^{*,w} \oplus \mathcal{D}_{(A,\Phi)}$. When Y is the standard S^3 , the virtual dimension of the moduli space "framed at infinity" $\tilde{\mathcal{M}} = \Theta^{-1}(0)/\mathcal{G}_0^w \subset \tilde{\mathcal{B}}^w$ is given by

$$\text{ind}^+(\mathcal{I}_\Phi^* \oplus \mathcal{D}_{(A,\Phi)}) + \dim \Gamma_\theta = d(c) + 1,$$

where $d(c)$ is in (2.7). The genuine moduli space is $\mathcal{M} = \tilde{\mathcal{M}}/\Gamma_\theta$ whose virtual dimension is $d(c)$. In general, \mathcal{M} and $\tilde{\mathcal{M}}$ are not smooth manifolds, and we need to perturb the equations. Before that, we introduce several terms.

Definition 4.16. The moduli space \mathcal{M} (or $\tilde{\mathcal{M}}$) is said to be *with vanishing i -th cohomology* if all of elements $[(A, \Phi)]$ of \mathcal{M} (or $\tilde{\mathcal{M}}$) have $H_{(A,\Phi)}^i = 0$. In particular,

- \mathcal{M} is said to be *regular* if all of elements $[(A, \Phi)]$ of \mathcal{M} have $H_{(A,\Phi)}^2 = 0$.
- \mathcal{M} is said to be *acyclic* if all of elements $[(A, \Phi)]$ of \mathcal{M} have $H_{(A,\Phi)}^k = 0$ for $k = 0, 1, 2$.

Remark 4.17. If \mathcal{M} contains no reducibles, then \mathcal{M} is with vanishing 0-th cohomology. But the converse is not necessarily true, because the stabilizer of a Pin⁻(2)-monopole reducible $[(A, 0)]$ on a twisted Spin^c-structure is $\{\pm 1\}$, and then $H_{(A,0)}^0 = 0$.

If $b_+(X; l) \geq 1$, by perturbing the equation by adding a *compactly-supported* self-dual 2-form as in (2.5), we obtain a smooth $\tilde{\mathcal{M}}$:

Theorem 4.18 ([7], Proposition 8.2.1). *Suppose $b_+(X; l) \geq 1$. For generic compactly-supported self-dual 2-forms, by perturbing the equations as in (2.5), the perturbed moduli space $\tilde{\mathcal{M}}$ is regular and contains no reducibles, and therefore is a smooth manifold of dimension $d(c) + 1$. Then \mathcal{M} is a smooth manifold of dimension $d(c)$.*

For the proof of Theorem 2.23, we will need *cut-down* moduli spaces as in [17], §4(viii). Let us define $\mathcal{M} \rightarrow \mathcal{A}/\mathcal{G}$ by $[(A, \Phi)] \mapsto [A]$. For $\alpha \in \mathcal{A}/\mathcal{G}$, the cut-down moduli space is

$$\mathcal{M}^C = \mathcal{M} \cap \pi^{-1}(\alpha).$$

Theorem 4.19. *Let X be a closed oriented connected 4-manifold with a twisted Spin^c -structure satisfying that $b_+(X; l) = 0$, $\tilde{c}_1(E)^2 = 0$ and the Dirac index is 0. For a generic choice of $\alpha \in \mathcal{A}/\mathcal{G}$ and a compactly-supported self-dual 2-form, the cut-down moduli space \mathcal{M}^C is regular, and therefore \mathcal{M}^C consists of one reducible point and finite number of irreducible points.*

Proof. The proof is similar to that in [17], §4(viii). Due to the noncompactness of X , we need to modify the following point: Note that \mathcal{M}^C contains a unique reducible class $[A(\mu)]$ for every choice of compactly-supported self-dual 2-form μ .

Claim. Let R be the set of compactly-supported self-dual 2-forms such that $D_{A(\mu)}$ is surjective, (and hence, of course, also injective). Then R is open-dense.

The proof of the claim is similar to that of Lemma 14.2.1 of [7]. With this understood, the rest of the proof of the theorem is similar to [17]. \square

5. Proofs of gluing formulae

The purpose of this section is to give proofs of the gluing formulae in §2(vi).

5(i). **Gluing monopoles.** Let X_1 and X_2 be Spin^c -4-manifolds with ends $[-1, \infty) \times Y_1$ and $[-1, \infty) \times Y_2$, where $Y_1 = \bar{Y}_2 = S^3$. Fix a reducible solution $(\theta, 0)$ on S^3 , and choose a C^∞ reference connection A_i^0 on each X_i which is the pull-back of θ on the tube. Let $x_i = (A_i, \Phi_i)$ be finite energy monopole solutions on X_i ($i = 1, 2$). Furthermore, we also suppose $H_{x_1}^2 = H_{x_2}^2 = 0$. We assume each A_i is in temporal gauge on the tube, and if necessary, consider it as a one-parameter family of connections $\theta + a_i(t)$ on the tube. The spinors Φ_i are also considered as one-parameter families $\Phi_i(t)$ on the tube.

Now, we construct an approximated solution on $X^{\#T}$ from (A_1, Φ_1) and (A_2, Φ_2) by splicing construction. Let us choose a smooth cut-off function γ , with $\gamma(t) = 1$ for $t \leq 0$ and $\gamma(t) = 0$ for $t \geq 1$. Let us define $x'_1 = (A'_1, \Phi'_1)$ over X_1 by

$$\begin{aligned} A'_1 &= \theta + \gamma(t - T + 3)a_1(t), \\ \Phi'_1 &= \gamma(t - T + 3)\Phi_1(t). \end{aligned}$$

We define $x'_2 = (A'_2, \Phi'_2)$ over X_2 in a similar fashion.

Fix an identification of the Spin^c -structures on $[0, 2T] \times Y_1$ and $[0, 2T] \times Y_2$ with respect to θ . Note that the Spin^c -structures on the tubes are untwisted, which can be considered as ordinary Spin^c -structures. The all possibilities of such identifications are parameterized

by Γ_θ , which are called the *gluing parameters*. If we fix an identification σ_0 , then the other identifications are indicated as $\sigma = \exp(v)\sigma_0$ for $v \in \text{Lie } \Gamma_\theta \cong i\mathbb{R}$. For an identification σ , we can glue x'_1 and x'_2 via σ to give a configuration over $X^{\#T}$. The glued configuration is denoted by

$$x'(\sigma) = (A'(\sigma), \Phi'(\sigma)).$$

Then it is easy to see the following

Proposition 5.1. *For each $i = 1, 2$, let Γ_i be the stabilizer of the monopole x_i . Then $x'(\sigma_1)$ and $x'(\sigma_2)$ are gauge equivalent if and only if $[\sigma_1] = [\sigma_2]$ in $\Gamma_\theta/(\Gamma_1 \times \Gamma_2)$, where Γ_i are the stabilizers of x_i .*

Let $\text{Gl} = \Gamma_\theta/(\Gamma_1 \times \Gamma_2)$. Let us define the map $J: \text{Gl} \rightarrow \mathcal{B}(X^{\#T})$ by the splicing construction above: $[\sigma] \mapsto [x'(\sigma)]$. If $H_{x_1}^2 = H_{x_2}^2 = 0$ and T is sufficiently large, then we can find in a unique way a monopole solution $x(\sigma)$ on $X^{\#T}$ near the spliced configuration $x'(\sigma)$. (The construction is explained in the Appendix.) Then we have a smooth map

$$(5.2) \quad I: \text{Gl} \rightarrow \mathcal{M}(X^{\#T}), \quad [\sigma] \mapsto [x(\sigma)].$$

Before proceeding, we give another description of the spliced family $\{[x'(\sigma)]\}$ for gluing parameters $\sigma \in \Gamma_\theta$. According to the definition of $x'(\sigma)$, for different σ , $x'(\sigma)$ are objects on different bundles parameterized by σ . It is convenient if we can represent all $[x'(\sigma)]$ as objects on a fixed identification, say σ_0 , of bundles. This is also done in [4], §7.2.4, in the ASD case.

Recall $X^{\#T} = X_1^0 \cup ([-T, T] \times Y) \cup X_2^0$, and X_1^{2T} and X_2^{2T} are assumed to be embedded in $X^{\#T}$. Choose a smooth function λ_1 on $X^{\#T}$ such that $\lambda_1 = 1$ on X_1^0 , $\lambda_1 = 0$ on X_2^0 and

$$\lambda_1(t, y) = \begin{cases} 1 & -T \leq t \leq -1, y \in Y, \\ 0 & 1 \leq t \leq T, y \in Y, \end{cases}$$

and satisfies $|\nabla \lambda| = O(1)$. Define another function λ_2 on $X^{\#T}$ by $\lambda_2 = 1 - \lambda_1$. Let $v \in \text{Lie } \Gamma_\theta = i\mathbb{R}$, and $\sigma = \sigma_0 \exp(v)$. Define gauge transformations h_1 and h_2 on $X^{\#T}$ by

$$(5.3) \quad \begin{aligned} h_1 &= \exp(\lambda_2 v) \\ h_2 &= \exp(-\lambda_1 v) \end{aligned}$$

Note that $h_1 h_2^{-1} = \exp(\lambda_1 + \lambda_2)v = \exp v$. Then $h_1 x'_1 = h_2 x'_2$ over $[-2, 2] \times Y$ on which x'_1 and x'_2 are flat, and therefore we can glue them. The glued configuration is denoted by $x'(\sigma_0, v)$. Then, by definition, it can be seen that $x'(\sigma)$ and $x'(\sigma_0, v)$ are gauge equivalent. Often, we will not distinguish these two, and use the same symbol $x'(\sigma)$.

5(ii). **Gluing map.** The gluing construction (5.2) can be globalized to whole moduli spaces. In fact, we can define the map

$$\Xi: \tilde{\mathcal{M}}(X_1) \times_{\Gamma_\theta} \tilde{\mathcal{M}}(X_2) \rightarrow \mathcal{M}(X^{\#T}),$$

for sufficiently large T . (See the Appendix.)

Theorem 5.4. *Let X_1 and X_2 be Spin^{c-} -4-manifolds with ends $[-1, \infty) \times Y_1$ and $[-1, \infty) \times Y_2$, where $Y_1 = \bar{Y}_2 = S^3$. (The Spin^{c-} -structures may be twisted or untwisted.) Suppose that $\mathcal{M}(X_i)$ are regular and $\mathcal{M}(X_1)$ contains no reducibles. Then there exists a large T such that the gluing map Ξ is a homeomorphism, and $\mathcal{M}(X^{\#T})$ is regular and contains no reducible. Furthermore, the following hold:*

- *Restricting to outside of the quotient singularities of reducibles, Ξ is a diffeomorphism.*
- *If each $\mathcal{M}(X_i)$ consists only of reducibles or irreducibles, then Ξ is a diffeomorphism.*

We note the following special case.

Corollary 5.5. *Suppose X_1 is a Spin^{c-} -4-manifold with the end $[-1, \infty) \times S^3$ whose moduli space $\mathcal{M}(X_1)$ is regular and contains no reducibles. Suppose X_2 is a Spin^c (untwisted Spin^{c-})-4-manifold with the end $[-1, \infty) \times S^3$ which is diffeomorphic to either of the following*

- *an open 4-ball with the standard Spin^c -structure, or*
- *$\overline{\mathbb{CP}}^2 \setminus D^4$ with the Spin^c -structure whose c_1 is a generator of $H^2(X_2; \mathbb{Z})$.*

Furthermore X_2 is supposed to be equipped with a metric whose scalar curvature is bounded below by a positive constant. Then Ξ induces a diffeomorphism

$$\mathcal{M}(X_1) \rightarrow \mathcal{M}(X^{\#T}).$$

The proofs for these are similar to those of the corresponding theorems in the Seiberg-Witten and Donaldson theory ([4, 3, 7] etc.). A proof based on [4, 3] will be explained in the Appendix.

5(iii). **The images of the map I .** To prove the gluing formulae, we want to know what is the homology class of the image of I in $H_*(\mathcal{B})$. The homology class depends on whether each of the Spin^{c-} -structures on X_1 and X_2 is twisted or untwisted, and whether each of monopoles x_1 and x_2 is irreducible or not. We call an irreducible/reducible monopole on a twisted Spin^{c-} -structure $\text{Pin}^-(2)$ -irreducible/reducible, and an irreducible/reducible monopole on an untwisted Spin^{c-} -structure $\text{U}(1)$ -irreducible/reducible. We assume that at least one of Spin^{c-} -structures of x_i is twisted. Then $\mathcal{B}(X^{\#T})$ is homotopy equivalent to $\mathbb{RP}^\infty \times T^{b_1(X^{\#T}; \mathbb{Z})}$. Let η be the generator of $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$. For monopoles x_1 and x_2 on X_1 and X_2 , let C be the image of I . Suppose x_1 and x_2 are not $\text{U}(1)$ -reducible. Then C is a circle.

Theorem 5.6. *For the homology class $[C] \in H_1(\mathcal{B}; \mathbb{Z})$ of C , we have the following:*

- (1) *If x_1 is a $\text{Pin}^-(2)$ -reducible and x_2 is a $\text{U}(1)$ -irreducible, then $\langle \eta, [C]_2 \rangle \neq 0$, where $[C]_2 \in H_1(\mathcal{B}^*; \mathbb{Z}_2)$ is the mod-2 reduction of $[C]$.*
- (2) *If x_1 is a $\text{Pin}^-(2)$ -irreducible and x_2 is a $\text{U}(1)$ -irreducible, then $[C] = [C]_2 = 0$.*
- (3) *If x_1 is a $\text{Pin}^-(2)$ -reducible and x_2 is a $\text{Pin}^-(2)$ -irreducible, $\langle t_0, [C] \rangle = \pm 1$ for a generator $t_0 \in H^1(T_0; \mathbb{Z})$. (See (2.22).)*
- (4) *If both of x_1 and x_2 are $\text{Pin}^-(2)$ -irreducibles, then $\langle t_0, [C] \rangle = \pm 2$.*

Before proving the theorem, we give some preliminaries. In the following, we simplify the notation as $\mathcal{G} = \mathcal{G}^w$, $\mathcal{G}_0 = \mathcal{G}_0^w$ and $\mathcal{K} = \mathcal{K}_\gamma$. Let $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{G}_0$. For each $i = 1, 2$, let \mathcal{S}_i be the set of solutions which are \mathcal{G} -equivalent to x_i . Now, we prove the assertions (1) and (2).

Proof of (1) and (2). We have a commutative diagram whose vertical and horizontal arrows are exact:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \uparrow & & \uparrow & & \\
 & & \{\pm 1\} & \equiv & \{\pm 1\} & & \\
 & & \uparrow & & \uparrow & & \\
 1 & \longrightarrow & \mathcal{G}_0 & \longrightarrow & \mathcal{G} & \longrightarrow & \Gamma_\theta \longrightarrow 1, \\
 & & \uparrow & & \uparrow & & \parallel \\
 1 & \longrightarrow & \mathcal{K}_0 & \longrightarrow & \mathcal{K} & \longrightarrow & \Gamma_\theta \longrightarrow 1, \\
 & & \uparrow & & \uparrow & & \\
 & & 1 & & 1 & &
 \end{array}$$

We also have the following diagrams of various quotient maps:

$$\begin{array}{ccc}
 & \mathcal{S}_1/\mathcal{K}_0 & \\
 \swarrow \{\pm 1\} & & \searrow \Gamma_\theta \\
 \mathcal{S}_1/\mathcal{G}_0 & & \mathcal{S}_1/\mathcal{K} \\
 \searrow \Gamma_\theta & & \swarrow \{\pm 1\} \\
 & \mathcal{S}_1/\mathcal{G} &
 \end{array}
 \qquad
 \begin{array}{c}
 \mathcal{S}_2/\mathcal{G}_0 \\
 \downarrow \Gamma_\theta \\
 \mathcal{S}_2/\mathcal{G}
 \end{array}$$

By definition, $\mathcal{S}_1/\mathcal{G}$ and $\mathcal{S}_2/\mathcal{G}$ are one-point sets. Then $\mathcal{S}_2/\mathcal{G}_0$ is a circle on which Γ_θ acts freely. Hence, $C = \text{Im } I$ can be written as

$$C = (\mathcal{S}_1/\mathcal{G}_0) \times_{\Gamma_\theta} (\mathcal{S}_2/\mathcal{G}_0) = \frac{(\mathcal{S}_1/\mathcal{K}_0) \times_{\Gamma_\theta} (\mathcal{S}_2/\mathcal{G}_0)}{\{\pm 1\}} = (\mathcal{S}_1/\mathcal{K}_0)/\{\pm 1\}.$$

First, let us consider the case of (2). In this case, \mathcal{G} acts on \mathcal{S}_1 freely. Therefore $\mathcal{S}_1/\mathcal{K}_0 \cong \Gamma_\theta \times \{\pm 1\}$, and we can see that the homology class of C is zero. In the case of (1), each element of \mathcal{S}_1 has the stabilizer $\{\pm 1\} \subset \mathcal{G}$. Since $\mathcal{G}_0 \cap \{\pm 1\} = \{1\}$, we see that $\mathcal{S}_1/\mathcal{G}_0 \cong \Gamma_\theta/\{\pm 1\}$ and $[C]$ is the generator of $H_1(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2$. \square

In order to prove the assertions (3) and (4), we first consider the gluing of connections. For each $i = 1, 2$, let A_i be a connection on the characteristic bundle E_i for c_i . Let us

consider spliced connections $A_1 \#_\sigma A_2$ on $E = E_1 \#_\sigma E_2$ as in §5(i), where $\sigma \in \Gamma_\theta$ are gluing parameters. Note that $A_1 \#_\sigma A_2$ is gauge equivalent to $A_1 \#_{-\sigma} A_2$, where $-\sigma = \sigma \exp \pi i$.

Lemma 5.7. *Let $S = \{[A_1 \#_\sigma A_2]\}_{\sigma \in \Gamma_\theta / \{\pm 1\}} \subset \mathcal{A}(E)/\mathcal{G}$ be the set of gauge equivalence classes of the family $\{A_1 \#_\sigma A_2\}_{\sigma \in \Gamma_\theta}$. Then S represents a primitive class in $H_1(\mathcal{A}(E)/\mathcal{G}; \mathbb{Z}) \cong \mathbb{Z}_2^{b_1^l}$. (The primitive class represented by S is the dual of t_0 .)*

Proof. Let us fix $\sigma_0 \in \Gamma_\theta$ as based point, and the spliced connections $A_1 \#_\sigma A_2$ for other σ are constructed by using (5.3) as in §5(i). For $\sigma \in \Gamma_\theta$, $A_1 \#_\sigma A_2$ and $A_1 \#_{-\sigma} A_2$ are gauge equivalent by the gauge transformation \check{g} such that

$$\check{g} = \begin{cases} 1 & \text{on } X_1^0 \\ -1 & \text{on } X_2^0 \\ \exp(\lambda_2 \pi i) & \text{on } [-T, T] \times Y \end{cases}$$

where λ_2 is the function defined around (5.3). On the other hand, for any w with $0 < w < \pi$, if we put $\sigma_w = \sigma \exp(iw)$, then $A_1 \#_\sigma A_2$ and $A_1 \#_{\sigma_w} A_2$ are not gauge equivalent. Therefore S is a circle embedded in $\mathcal{A}(E)/\mathcal{G}$. By taking homotopy class and projection, we have a surjection $\rho: \mathcal{G} \rightarrow H^1(X; l)/\text{Tor}$ (see [17], Lemma 4.22). Since $\mathcal{A}(E)/\mathcal{G}$ is isomorphic to the Picard torus $H^1(X; l \otimes \mathbb{R})/(H^1(X; l)/\text{Tor})$, it suffices to prove $\rho(\check{g})$ is a primitive element in $H^1(X; l)/\text{Tor}$. To see this, let us consider the following commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{G}} = \text{Map}(\tilde{X}; \text{U}(1)) & \xrightarrow{\tilde{\rho}} & H^1(\tilde{X}; \mathbb{Z}) \\ \varpi \uparrow & & \varpi' \uparrow \\ \mathcal{G} = \Gamma(\tilde{X} \times_{\{\pm 1\}} \text{U}(1)) & \xrightarrow{\rho} & H^1(X; l)/\text{Tor}, \end{array}$$

where the maps ϖ and ϖ' are the pull-back maps to the double covering \tilde{X} . Note the following:

- The image of ϖ is the fixed point set $\tilde{\mathcal{G}}^I$, where the I -action is given by $I\check{g} = \overline{\iota^* \check{g}}$.
- Let \tilde{X}_i ($i = 1, 2$) be the double coverings of X_i . Then \tilde{X} is the connected sum “at two points” of \tilde{X}_1 and \tilde{X}_2 . That is, this is obtained as follows: For each $i = 1, 2$, removing two 4-balls from each of \tilde{X}_i , we obtain a manifold \tilde{X}'_i whose boundary \tilde{Y}_i is a disjoint union of two S^3 . Then $\tilde{X} = \tilde{X}'_1 \cup_{\tilde{Y}_1 = \tilde{Y}_2} \tilde{X}'_2$.

To see that $\rho(\check{g})$ is primitive, we claim that $\tilde{\rho} \circ \varpi(\check{g})$ is a primitive element in $H^1(\tilde{X}; \mathbb{Z})$. To see this, let us consider a circle γ embedded in \tilde{X} starting from a point x_1 in \tilde{X}'_1 and entering \tilde{X}'_2 via a component of $\tilde{Y}_1 = \tilde{Y}_2$ and returning to x_1 via another component of $\tilde{Y}_1 = \tilde{Y}_2$. Then the restriction of $\varpi(\check{g})$ to γ gives a degree one map from γ to $\text{U}(1)$. \square

Proof of (3) and (4). Let us consider the projection

$$\pi: C = (\mathcal{S}_1/\mathcal{G}_0) \times_{\Gamma_\theta} (\mathcal{S}_2/\mathcal{G}_0) \rightarrow S,$$

which is defined by $\pi([x_1], [x_2]) = [A_1 \#_\sigma A_2]$, where each A_i is the connection part of x_i . Note that π is a map between two S^1 . Then, π has degree 1 in the case of (3), and degree 2 in the case (4). \square

5(iv). Proofs of gluing formulae.

Proof of Theorem 2.23. For each $i = 1, 2$, let X'_i be the manifold with cylindrical end obtained from removing a 4-ball from X_i . By perturbing the equations with a compactly-supported 2-form, Corollary 5.5 implies that $\tilde{\mathcal{M}}(X_1) \cong \tilde{\mathcal{M}}(X'_1)$ for a metric on X_1 with long neck. By the assumption, $\mathcal{M}(X'_1)$ consists of odd numbers, say k , of U(1)-irreducible points. First let us consider the case when $b_1^l = b_1(X_2; l_2) = 0$. In this case, $\mathcal{M}(X_2)$ consists of one Pin⁻(2)-reducible point and maybe several Pin⁻(2)-irreducible points. By Theorem 5.4, $\mathcal{M}(X^{\#T})$ is a disjoint union of several circles:

$$\mathcal{M}(X^{\#T}) = \bigcup_{i=1}^k C_i \cup \bigcup_j C'_j,$$

where C_i are obtained by gluing U(1)-irreducibles and a Pin⁻(2)-reducible, and C'_j are made from U(1)-irreducibles and Pin⁻(2)-irreducibles. Then Theorem 5.6 implies that $\langle \eta, [\mathcal{M}(X^{\#T})] \rangle = k \bmod 2$, and this implies the theorem.

In the case when $b_1^l \geq 1$, by using the cut-down moduli space $\mathcal{M}^C(X_2)$ in Theorem 4.19 instead of $\mathcal{M}(X_2)$, the argument is reduced to the case when $b_1^l = 0$. \square

Proof of Theorem 2.24. Obvious from Corollary 5.5. \square

Proof of Theorem 2.25. For simplicity, we consider the case when $b_1(X_2; l_2) = 0$. The generalization to the other cases will be clear. By Theorem 5.4, we have an identification

$$\mathcal{M}(X^{\#T}) \cong \tilde{\mathcal{M}}(X_1) \times_{\Gamma_\theta} \tilde{\mathcal{M}}(X_2).$$

By Theorem 5.6, we see that

$$\langle \xi, [\mathcal{M}(X^{\#T})] \cap t_0 \rangle = \langle \xi, [\mathcal{M}(X_1)] \rangle.$$

This implies the theorem. \square

Proof of Theorem 2.26. For simplicity, suppose $d(c_1) = 0$. The general case will be obvious. The case (1) is proved by Theorem 5.6(2). In the case (2), $\mathcal{M}(X^{\#T})$ is a disjoint union of circles C_i , such that $\langle t_0, [C_i] \rangle = \pm 2$. Therefore \mathbb{Z}_2 -valued invariant is zero. \square

By the proof of the case (2) of Theorem 2.26, Theorem 2.27 is true if the glued moduli is orientable. The orientability of the glued moduli space follows from the next lemma:

Lemma 5.8. *For $i = 1, 2$, let X_i be a closed oriented connected 4-manifold with a twisted Spin^c-structure c_i whose Dirac index is positive and even, and A_i be a connection on the characteristic bundle E_i . Then for S in Lemma 5.7, the restriction of $\text{ind } \delta_{\text{Dirac}}$ to S , $\text{ind}(\delta_{\text{Dirac}}|_S)$, is orientable.*

Proof. We construct a framing of the index bundle $\text{ind}(\delta_{\text{Dirac}}|_S)$. For simplicity, we assume $\text{ind } D_{A_2} = 2$, and the general case will be clear. Let us consider the family $\{D_{A_1 \#_\sigma A_2}\}_{\sigma \in \Gamma_\theta}$. By Proposition 2.2 in [1], we may assume $\text{Coker } D_{A_1 \#_\sigma A_2} = 0$ for any σ . Since $\ker D_\theta = 0$ on S^3 , we can construct an isomorphism for each σ ([3], §3.3):

$$\alpha_\sigma: \ker D_{A_1} \oplus \ker D_{A_2} \rightarrow \ker D_{A_1 \#_\sigma A_2}.$$

In the proof of Lemma 5.7, we have seen that $A_1 \#_{\sigma} A_2$ is gauge equivalent to $A_1 \#_{-\sigma} A_2$ by a gauge transformation g . Now we can see that, for $\psi \in \text{Ker } D_{A_1}$ and $\phi \in \text{Ker } D_{A_2}$,

$$\alpha_{\sigma}(\psi, \phi) = g\alpha_{-\sigma}(\psi, -\phi).$$

Let $\{\psi^j\}$ be a basis for $\text{ker } D_{A_1}$, and $\{\phi^1, \phi^2\}$ be a basis for $\text{ker } D_{A_2}$. Fix $\sigma_0 \in \Gamma_{\theta}$ and let $\sigma_w = \sigma_0 \exp(iw)$ for $0 \leq w \leq \pi$, and

$$\begin{pmatrix} \phi_w^1 \\ \phi_w^2 \end{pmatrix} = \begin{pmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{pmatrix} \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}.$$

Then the following gives a framing for $\text{ind}(\delta_{\text{Dirac}}|_S)$:

$$\{\alpha_{\sigma_w}(\psi^j, \phi_w^1), \alpha_{\sigma_w}(\psi^j, \phi_w^2)\}.$$

□

Corollary 5.9. *For each $i = 1, 2$, let X_i be a closed oriented connected 4-manifold with a twisted Spin^c -structure which has the following properties:*

- *the index of the Dirac operator is positive and even, and*
- *the moduli space $\mathcal{M}(X_i)$ is orientable.*

Then the glued moduli space $\mathcal{M}(X_1 \# X_2)$ is also orientable.

Proof of Theorem 2.27. Since each of $\mathcal{M}(X_i)$ is orientable, Corollary 5.9 implies $\mathcal{M}(X_0 \# \cdots \# X_n)$ is also orientable. The statement for the invariant is proved by Theorem 5.6. □

6. Proofs of Theorem 1.19 and Corollary 1.5

In this section, we prove Theorem 1.19 and Corollary 1.5. Our proof of Theorem 1.19 is similar to the proof of Thom conjecture due to Kronheimer and Mrowka [12]. (Cf. [18].) We begin with the proof of Theorem 1.19.

6(i). **Reduction to the case when $\alpha \cdot \alpha = 0$.** Suppose $n := \alpha \cdot \alpha > 0$. Let $X' = X \# n \overline{\mathbb{CP}}^2$, and E_i ($i = 1, \dots, n$) be the (-1) -sphere in the i -th $\overline{\mathbb{CP}}^2$. Take the connected sum in X' ,

$$\Sigma' = \Sigma \# E_1 \# \cdots \# E_n.$$

Then $[\Sigma'] \cdot [\Sigma'] = 0$.

Even if we replace X by X' , the $\text{Pin}^-(2)$ -monopole invariant is unchanged by Theorem 2.24. Furthermore, even if we replace \tilde{X} by \tilde{X}' , the Seiberg-Witten invariant is also unchanged by the ordinary blow-up formulae [5, 18]. The quantity $-\chi(\Sigma)$ and $\alpha \cdot \alpha + |\tilde{c}_1(E) \cdot \alpha|$ are also unchanged. Thus, we may assume $\alpha \cdot \alpha = 0$.

6(ii). **The case when $\chi(\Sigma) > 0$.** Here, we point out that, under the assumption of Theorem 1.19, the Euler number of Σ cannot be positive:

Proposition 6.1. *Suppose (X, α, Σ) satisfies the assumption of the beginning of §1(ii). If we suppose the following:*

- $\chi(\Sigma) > 0$,
- $\alpha = [\Sigma] \in H_2(X; l)$ has an infinite order, and

- $\alpha \cdot \alpha = 0$,

then the Pin⁻(2)-monopole invariants of (X, c) and the Seiberg-Witten invariants of (\tilde{X}, \tilde{c}) are trivial.

Proof. The Seiberg-Witten case is proved by Theorem 1.1.1 in [7] or Proposition 4.6.5 in [18]. The Pin⁻(2)-monopole case is similar. Take a tubular neighborhood N of Σ , and let $Y = \partial N$ and $X_0 = \overline{X \setminus N}$. Then Y admits a positive scalar curvature metric g_Y . Decompose X as $X = X_0 \cup_Y N$. For a positive real number T , let us insert a cylinder between X_0 and N as:

$$X_T = X_0 \cup ([-T, T] \times Y) \cup N.$$

Fix a metric on X_T which is product on the cylinder: $dt^2 + g_Y$. By the assumption, $\alpha \otimes \mathbb{R}$ is a nonzero class in $H_2(X_T; \lambda)$. Let $a \in H^2(X_T; \lambda)$ be its Poincaré dual. Then the image of a by the restriction map $r: H^2(X_T; \lambda) \rightarrow H^2(Y; i^*\lambda)$ is also a nonzero class. Choose a 2-form $\eta \in \Omega^2(Y; i^*\lambda)$ representing $r(a)$. Let us perturb the Pin⁻(2)-monopole equations on Y by η as in (3.3). Since every Pin⁻(2)-monopole solution for a positive scalar curvature metric g_Y is reducible, a generic choice of η makes the perturbed Chern-Simons-Dirac functional (3.6) have no critical point. Choose a 2-form $\mu \in i\Omega^2(X; \lambda)$ whose restriction to the cylinder is the pull-back of $i\eta$.

Now suppose the Pin⁻(2)-monopole invariants of (X, c) is nontrivial. Then the moduli space $\mathcal{M}(X_T)$ is nonempty for all T . Taking the limit $T \rightarrow \infty$, we can obtain a finite energy solution on the manifold with cylindrical end, $X_0 \cup [-1, \infty) \times Y$. Since a finite energy solution should converge to a critical point at infinity (Corollary 4.9), this is a contradiction. \square

6(iii). **The case when Σ is nonorientable.** Take a tubular neighborhood N of Σ , and let $Y = \partial N$ and $X_0 = \overline{X \setminus N}$. Decompose X as $X = X_0 \cup_Y N$. For a large $T > 0$, let us insert a long cylinder between X_0 and N as:

$$X_T = X_0 \cup ([-T, T] \times Y) \cup N.$$

Fix a metric on X_T which is product on the cylinder: $dt^2 + g_Y$. (Below, we will take a special metric g_Y on Y .) Let \tilde{X}_T be the associated double covering. Then

$$\tilde{X}_T = \tilde{X}_0 \cup ([-T, T] \times \tilde{Y}) \cup \tilde{N},$$

where $\tilde{Y} = S^1 \times \tilde{\Sigma}$ and $\tilde{N} = D^2 \times \tilde{\Sigma}$. (The object with \sim is the associated double covering.) Let us take the metric g_Y on Y so that its pull-back metric on $\tilde{Y} = S^1 \times \tilde{\Sigma}$ is of the form

$$d\theta^2 + g_{\tilde{\Sigma}},$$

where $g_{\tilde{\Sigma}}$ is the metric with constant scalar curvature $-2\pi(4g(\tilde{\Sigma}) - 4)$. Then the volume of $\tilde{\Sigma}$ is 1.

Now, let us consider the limit $T \rightarrow \infty$. For \tilde{X}_T , the following is known.

Proposition 6.2 ([12]). *If the Seiberg-Witten invariant of (\tilde{X}, \tilde{c}) is nontrivial, then there is a translation invariant Seiberg-Witten solution on $\mathbb{R} \times \tilde{Y}$.*

Similarly, we can prove the following:

Proposition 6.3. *If the $\text{Pin}^-(2)$ -monopole invariant of (X, c) is nontrivial, then there exists a translation invariant $\text{Pin}^-(2)$ -monopole solution on $\mathbb{R} \times Y$.*

Under the situation of Proposition 6.3, by pulling back the $\text{Pin}^-(2)$ -monopole solution on $\mathbb{R} \times Y$ to $\mathbb{R} \times \tilde{Y}$, we also have a translation invariant Seiberg-Witten solution on $\mathbb{R} \times \tilde{Y}$.

By the argument in [12], the existence of a translation invariant solution on $\mathbb{R} \times \tilde{Y}$ implies

$$-\chi(\tilde{\Sigma}) \geq |c_1(L)[\tilde{\Sigma}]|.$$

This immediately implies

$$-\chi(\Sigma) \geq |\tilde{c}_1(E)[\Sigma]|.$$

6(iv). **The case when Σ is orientable.** In the case when the $\text{Pin}^-(2)$ -monopole invariant of (X, c) is nontrivial, let us consider X_T with long neck. Since the restriction of the local system l to Σ is trivial for orientable Σ , the $\text{Pin}^-(2)$ -monopole equations on N and Y are in fact the Seiberg-Witten equations. This reduces the argument to the Seiberg-Witten case [12]. Let us consider the case when the Seiberg-Witten invariant of (\tilde{X}, \tilde{c}) is nontrivial. Since Σ is orientable, $\tilde{\Sigma}$ has two components: $\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$. Then take a tubular neighborhood \tilde{N}_1 of $\tilde{\Sigma}_1$, and let $\tilde{Y}_1 = \partial\tilde{N}_1$ and $\tilde{X}_0 = \tilde{X} \setminus \tilde{N}_1$. Let us consider

$$\tilde{X}'_T = \tilde{X}_0 \cup ([-T, T] \times \tilde{Y}_1) \cup \tilde{N}_1,$$

for large T . This also reduces the argument to the Seiberg-Witten case [12].

Proof of Corollary 1.5. Since $(\iota_*)^2 = \text{id}$, $H_2(\tilde{X}; \mathbb{Q})$ splits into (± 1) -eigenspaces. Then (-1) -eigenspace is identified with $H_2(X; l \otimes \mathbb{Q})$. Let $\pi: \tilde{X} \rightarrow X$ be the projection. Then $\pi_*: H_2(\tilde{X}; \mathbb{Q}) \rightarrow H_2(X; l \otimes \mathbb{Q})$ can be identified with $\alpha \mapsto \frac{1}{2}(\alpha - \iota_*\alpha)$. It follows from these and the assumption that $\Sigma \cap \iota\Sigma = \emptyset$ that $\pi(\Sigma)$ satisfies the conditions in Theorem 1.3. \square

7. A discussion on the genus estimate

Let K be a $K3$ surface. Let us recall a result due to C. T. C. Wall.

Theorem 7.1 (Wall[22]). *Every primitive class in $H_2(K \# (S^2 \times S^2); \mathbb{Z})$ is represented by an embedded S^2 .*

On the other hand, Theorem 1.19 implies the following:

Corollary 7.2. *For $K \# (S^2 \times \Sigma_g)$ with positive g and the \mathbb{Z} -bundle l over $S^2 \times \Sigma_g$ as in Theorem 1.12, a primitive class α in $H_2(K \# (S^2 \times \Sigma_g); \mathbb{Z} \# l)$ of an infinite order with $\alpha \cdot \alpha \geq 0$ cannot be represented by an embedded S^2 .*

Remark 7.3. Note that $H_2(K \# (S^2 \times \Sigma_g); \mathbb{Z} \# l_T) \cong H_2(K; \mathbb{Z}) \oplus \mathbb{Z}/2$.

By the adjunction inequality obtained by the ordinary Seiberg-Witten theory [15] (Cf. [12, 5, 20, 18]), a primitive class α in $H_2(K; \mathbb{Z})$ with $\alpha \cdot \alpha \geq 0$ cannot be represented by an embedded S^2 . Let K' be a manifold obtained by removing an embedded 4-ball from a $K3$ surface K . Note that the inclusion $K' \hookrightarrow K$ induces an isomorphism on the

second homology groups. We may assume K' is embedded in both of $K\#(S^2 \times \Sigma_g)$ and $K\#(S^2 \times S^2)$. Let us consider the following injective homomorphisms:

$$\begin{aligned}\phi_0: H_2(K; \mathbb{Z}) &\cong H_2(K'; \mathbb{Z}) \rightarrow H_2(K\#(S^2 \times S^2); \mathbb{Z}), \\ \phi_1: H_2(K; \mathbb{Z}) &\rightarrow H_2(K\#(S^2 \times \Sigma_g); \mathbb{Z}), \\ \phi'_1: H_2(K; \mathbb{Z}) &\rightarrow H_2(K\#(S^2 \times \Sigma_g); \mathbb{Z}\#l).\end{aligned}$$

Corollary 7.2 immediately implies every primitive class in $\text{Im } \phi'_1$ with nonnegative self-intersection cannot be realized by an embedded S^2 . On the other hand, Wall's theorem tells us that even if a primitive class with nonnegative self-intersection is in $\text{Im } \phi_0$, it can be realized by an embedded S^2 . As for primitive classes in $\text{Im } \phi_1$, we can prove the following by using Corollary 7.2.

Proposition 7.4. *If a primitive class $\alpha \in \text{Im } \phi_1$ satisfies $\alpha \cdot \alpha \geq 0$, then α cannot be realized by an embedded S^2 .*

Let us prove Proposition 7.4 The proof presented here is due to M. Furuta. Let us decompose $K\#T$ into $K' \cup_Y T'$, where $K' = K \setminus D^4$, $T' = T \setminus D^4$ and $Y = \partial K' = -\partial T'$. Suppose a primitive class $\alpha \in \text{Im } \phi_1$ is represented by an embedded 2-sphere. Take such an embedded 2-sphere Σ which is transverse to Y . Let $Z = K' \cup \Sigma$.

Lemma 7.5. *The restriction of $\mathbb{Z}\#l_T$ to Z is trivial.*

Proof. Let $C = Y \cap \Sigma$. Then C is a disjoint union of finite numbers of simple closed curves on Σ . Let $D = \Sigma \cap T'$. Then D is a domain on Σ and $\partial D = C$. Since $C \subset Y$, the restriction of $\mathbb{Z}\#l_T$ to C is trivial. Since $H_1(C; \mathbb{Z}) \rightarrow H_1(D; \mathbb{Z})$ is surjective, the restriction of l to D is also trivial. \square

Then Σ represents the corresponding class in the image of $H_2(K'; \mathbb{Z}) \rightarrow H_2(Z; \mathbb{Z})$. If we consider the composition of maps $H_2(K'; \mathbb{Z}) \rightarrow H_2(Z; \mathbb{Z}) \rightarrow H_2(K\#T; \mathbb{Z}\#l_T)$, a primitive class in $\text{Im } \phi'_1$ should be represented by an embedded S^2 . This contradicts with Theorem 1.3.

A. Appendix

The purpose of this appendix is to give a proof of Theorem 5.4. The proof is based on [4], §7.2 and [3], Chapter 4.

A(i). **The construction of the map I .** First, we give the construction of the map I of (5.2). Let $x_i = (A_i, \Phi_i)$ be finite energy monopole solutions on X_i ($i = 1, 2$). Fix a gluing parameter $\sigma_0 \in \Gamma_\theta$. Let $x'_0 = x'(\sigma_0)$ be the spliced configuration as in §5(i). The goal is to find a true solution $x(\sigma_0)$ near x'_0 under the assumptions $H_{x_1}^2 = H_{x_2}^2 = 0$.

The monopole map on $X^{\#T}$ is defined as a map between weighted spaces:

$$\Theta: L_1^{p, w_T} \rightarrow L^{p, w_T}.$$

Since the monopole solutions x_1 and x_2 decay exponentially (§4(ii)), we have an estimate

$$\|\Theta(x'_0)\|_{L^p} = O(e^{-\delta_0 T}).$$

Therefore we also have

$$(A.1) \quad \|\Theta(x'_0)\|_{L^{p,w_T}} = O(e^{(\alpha-\delta_0)T}).$$

Now we assume $\alpha < \delta_0$ so that the quantity above will be small for large T , and set

$$(A.2) \quad \delta = \delta_0 - \alpha.$$

We want to solve the equation for $y = (a, \phi) \in L_1^{p,w_T}(\Lambda^1(i\lambda) \oplus S^+)$

$$\Theta(x'_0 + y) = 0.$$

This equation is equivalent to

$$(A.3) \quad (\mathcal{D}_{x'_0} + n)(y) = -\Theta(x'_0),$$

where $\mathcal{D}_{x'_0}$ is the linearization of Θ , and n is the quadratic term:

$$n(y) = (q(\phi), \rho(a)\phi).$$

To solve (A.3), we first solve the linear version of it. For this purpose, we construct the right inverse Q of the linear operator $\mathcal{D}_{x'_0}$. The operator Q will be constructed by splicing the right inverses Q_1 and Q_2 for the linearizations of Θ , \mathcal{D}_{x_1} and \mathcal{D}_{x_2} , over X_1 and X_2 . First, we have the following:

Proposition A.4 ([3], §3.3). *For each monopole solutions x_i ($i = 1, 2$), if $H_{x_i}^2 = 0$, then there exists the right inverse Q_i for D_{x_i} : That is, there exists a map $Q_i: L^{p,w} \rightarrow L_1^{p,w}$ and a constant C_i which satisfy:*

$$D_{x_i} \circ Q_i(u) = u, \quad \|Q_i u\|_{L_1^{p,w}} \leq C_i \|u\|_{L^{p,w}},$$

for every $u \in L^{p,w}$.

The proof is a simple adaptation of the argument due to Donaldson[3], §3.3.

Now, an approximate inverse Q' for $\mathcal{D}_{x'_0}$ is constructed by splicing as follows: Recall $X^{\#T}$ is considered as the union $X^{\#T} = X_1^T \cup X_2^T$. Let $\chi_1: X^{\#T} \rightarrow \mathbb{R}$ be the characteristic function of X_1^T , that is

$$\chi_1(x) = \begin{cases} 1 & x \in X_1^T, \\ 0 & x \in X^{\#T} \setminus X_1^T. \end{cases}$$

Choose the function γ_1 such that

- $\gamma_1 = 1$ over the support of χ_1 ,
- the support of γ_1 is in $X_1^0 \cup [-T, T] \times Y$, and
- $|\nabla \gamma_1| = O(T^{-1})$.

Take χ_2 and γ_2 symmetrically. Then we have $\gamma_1 \chi_1 + \gamma_2 \chi_2 = 1$ everywhere. Now define

$$Q'(u) = \gamma_1 Q_1(\chi_1 u) + \gamma_2 Q_2(\chi_2 u).$$

Note that the w_T -norm of $\chi_1 u$ is equal to the w_1 -norm of that since the weight functions are equal on its support. Thus we have

$$\|Q_1(\chi_1 u)\|_{L_1^{p,w_T}} \leq C_1 \|u\|_{L^{p,w_T}},$$

where C_1 is the constant in Proposition A.4.

Recall that x_1 and x_2 are monopole solutions on X_1 and X_2 , x'_1 and x'_2 are configurations flattened on the ends and x'_0 is the spliced configuration on $X^{\#T}$. Therefore the linearization $\mathcal{D}_{x'_0}$ is equal to $\mathcal{D}_{x'_i}$ on the support of γ_i . Then

$$\begin{aligned} D_{x'_0} Q' u &= D_{x'_0} (\gamma_1 Q_1(\chi_1 u) + \gamma_2 Q_2(\chi_2 u)) \\ &= \gamma_1 \mathcal{D}_{x'_1} Q_1(\chi_1 u) + \gamma_2 \mathcal{D}_{x'_2} Q_2(\chi_2 u) + \nabla \gamma_1 * Q_1(\chi_1 u) + \nabla \gamma_2 * Q_2(\chi_2 u), \end{aligned}$$

where $*$ means an algebraic multiplication. Let us estimate each term of the last equation. The w_T -norm of $\nabla \gamma_1 * Q_1(\chi_1 u)$ is less than w_1 -norm of it since w_T is smaller than w_1 . Therefore

$$\|\nabla \gamma_1 * Q_1(\chi_1 u)\|_{L^{p,w_T}} \leq \text{const.} T^{-1} \|u\|_{L^{p,w_T}}.$$

Next we want to estimate $\gamma_1 \mathcal{D}_{x'_1} Q_1(\chi_1 u)$. The operator Q_1 is not the right inverse for $\mathcal{D}_{x'_1}$, but is that of \mathcal{D}_{x_1} . Since x_1 decay exponentially, the operator norm of the difference of these two is estimated as

$$\|\mathcal{D}_{x_1} - \mathcal{D}_{x'_1}\|_{\text{OP}} = O(e^{-\delta T}).$$

Then

$$\|(\text{id} - \mathcal{D}_{x'_1} \circ Q_1)u\|_{L^{p,w_1}} \leq \|\mathcal{D}_{x_1} - \mathcal{D}_{x'_1}\|_{\text{OP}} \cdot \|Q_1\|_{L^{p,w_1}} \leq \text{const.} e^{-\delta T} \|u\|_{L^{p,w_1}}.$$

Summing up these, we obtain

$$\|(\text{id} - \mathcal{D}_{x'_0} \circ Q)u\|_{L^{p,w_T}} \leq \text{const.} (e^{-\delta T} + T^{-1}) \|u\|_{L^{p,w_T}} \leq CT^{-1} \|u\|_{L^{p,w_T}},$$

for some constant C . If we take a large T so that $CT^{-1} < 1$, we obtain the inverse $(\mathcal{D}_{x'_0} Q')^{-1}$ by iteration. Then the true right inverse Q for $\mathcal{D}_{x'_0}$ is given by

$$Q = Q'(\mathcal{D}_{x'_0} Q')^{-1}.$$

For summary,

Proposition A.5. *There exists the operator $Q: L^{p,w_T} \rightarrow L^{p,w_T}$ which satisfies*

$$(\mathcal{D}_{x'_0} \circ Q)u = u, \quad \|Qu\|_{L^{p,w_T}} \leq \text{const.} \|u\|_{L^{p,w_T}},$$

for every $u \in L^{p,w_T}$.

Now we begin to seek the solution for (A.3). The main tool for this is the contraction mapping principle. We seek the solution of the form $y = Qu$. So to solve is

$$(A.6) \quad u + n(Qu) = -\Theta(x'_0).$$

We slightly change the function spaces. In (4.11), p is chosen so that $2 < p < 4$. Let q be the number defined by

$$(A.7) \quad 1 - \frac{4}{p} = -\frac{4}{q}.$$

By the Sobolev embedding, L^{p,w_T}_1 is continuously embedded in L^{q,w_T} . Set $U = L^{p,w_T}$ and $V = L^{q,w_T}$. Then the operator Q can be considered as a map from U to V with

$$(A.8) \quad \|Qu\|_V \leq C\|u\|_U,$$

for some constant C independent of T . Since n is a quadratic map, by using Hölder's inequality with the relation (A.7), we have an estimate that there is a constant M such that

$$(A.9) \quad \|n(y_1) - n(y_2)\|_U \leq M\|y_1 - y_2\|_V(\|y_1\|_V + \|y_2\|_V),$$

for any y_1, y_2 in V . Then we have

$$\|n(Qu_1) - n(Qu_2)\|_U \leq MC^2\|u_1 - u_2\|_U(\|u_1\|_U + \|u_2\|_U),$$

where the constants C and M are those in (A.8) and (A.9). Now if, for instance, $\|\Theta(x'_0)\|_U \leq (100MC^2)^{-1}$, then there exists a unique solution to (A.6). By (A.1), $\|\Theta(x'_0)\|_U$ can be arbitrary small if we take a sufficient large T . Thus for large T , we have a unique solution u . Let $y = Qu$. Then $x'_0 + y$ is a required monopole solution which is in L_1^{p,w_T} , and hence in C^∞ .

Thus for each $\sigma \in \Gamma$, we can find a monopole solution $x(\sigma)$ in a unique way near the spliced configuration $x'(\sigma)$. The correspondence $\sigma \mapsto x(\sigma)$ descends to the map I .

A(ii). **The image of I .** We would like to characterize the image of I . Let d be the metric on $\mathcal{B}(X^{\#T})$ given by

$$d([x], [y]) = \inf_{g \in \mathcal{G}} \|x - gy\|_V,$$

where $V = L^{q,w_T}$. For $\varepsilon > 0$, let $\mathcal{U}(\varepsilon) \subset \mathcal{B}(X^{\#T})$ be the open set

$$(A.10) \quad \mathcal{U}(\varepsilon) = \{[x] \mid d([x], J(\Gamma_\theta)) < \varepsilon, \|\Theta(x)\|_U < \varepsilon\}.$$

Proposition A.11. *If $H_{x_i}^0 = H_{x_i}^1 = H_{x_i}^2 = 0$ for $i = 1, 2$, then for small enough ε there exists $T(\varepsilon)$ so that for $T > T(\varepsilon)$ any point in $\mathcal{U}(\varepsilon)$ can be represented by a configuration of the form $x'(\sigma) + Q_\sigma u$ with $\|u\|_U < \text{const} \cdot \varepsilon$, where Q_σ is the right inverse for $D_{x'(\sigma)}$.*

Assuming the proposition, we have

Corollary A.12. *If $H_{x_i}^0 = H_{x_i}^1 = H_{x_i}^2 = 0$ for $i = 1, 2$, then for ε and $T(\varepsilon)$ in Proposition A.11, and for every $T > T(\varepsilon)$, the intersection $\mathcal{U}(\varepsilon) \cap \mathcal{M}(X^{\#T})$ is equal to the image of $I: \Gamma_\theta \rightarrow \mathcal{M}$.*

The corollary follows from the argument in §A(i) since under the given assumptions there is a unique small solution u to the equation $\Theta(x'(\sigma) + Q_\sigma u) = 0$.

A(iii). **Proof of Proposition A.11: Closedness.** Let us begin to prove Proposition A.11. The proposition is proved by continuity method. Let $[y]$ be an element of $\mathcal{U}(\varepsilon)$. Then there exists $x' \in J(\Gamma_\theta)$ with $\|x' - y\|_V < \varepsilon$. Let us write $y = x' + b$ and consider the path for $t \in [0, 1]$,

$$y_t = x' + tb.$$

By gauge transformation, we may assume y and b are smooth, and so is y_t for all t . It can be seen that, for given ε , if we take T large enough, the class $[y_t]$ is in $\mathcal{U}(\varepsilon)$ for every $t \in [0, 1]$. Let us define the subset $S \subset [0, 1]$ as the set of t which has the property that there exist $g_t \in \mathcal{G}$, $x'_{\sigma_t} \in J(\Gamma_\theta)$ and $u_t \in U = L^{p,w_T}$ such that

$$(A.13) \quad g_t y_t = x'_{\sigma_t} + Q_{\sigma_t}(u),$$

with $\|u\|_U < \nu$, where ν will be chosen below.

Obviously $0 \in S$. We would like to prove S is open and closed.

Let us prove the closedness. Suppose $t \in S$. Then there exist g_t, x'_{σ_t} and u_t so that (A.13) holds. Applying Θ on both sides of (A.13), we have

$$(A.14) \quad \Theta(g_t y_t) = \Theta(x'_{\sigma_t}) + \mathcal{D}_{\sigma_t} Q_{\sigma_t} u_t + n(Q_{\sigma_t} u_t) = \Theta(x'_{\sigma_t}) + u_t + n(Q_{\sigma_t} u_t).$$

Then we have an estimate

$$\|u_t\|_U \leq \|\Theta(y_t)\|_U + \|\Theta(x'(\sigma_t))\|_U + \|n(Q_{\sigma_t} u_t)\|_U \leq \varepsilon + \text{const.} e^{-\delta T} + (C_{\sigma_t})^2 \|u_t\|_U^2,$$

where C_{σ_t} is the constant for Q_{σ_t} so that $\|Q_{\sigma_t} u\| \leq \|u\|$. Since Γ_θ is compact, C_σ ($\sigma \in \Gamma_\theta$) is bounded above by some constant N as

$$(A.15) \quad C_\sigma \leq N.$$

Rearranging this and taking ν so that $\|u\| \leq \nu \leq (2N^2)^{-1}$, we have

$$(A.16) \quad \frac{1}{2} \|u_t\|_U \leq (1 - N^2 \|u_t\|) \|u_t\| \leq \varepsilon + \text{const.} e^{-\delta T}.$$

This estimate implies the following:

Lemma A.17. *Suppose $\nu \leq (2N^2)^{-1}$ so that the estimate (A.16) holds. Then we can choose small ε and large T so that $\|u\| < \nu$ implies $\|u\| \leq \frac{1}{2}\nu$.*

Thus the open condition $\|u\| < \nu$ is also closed.

Suppose we have $t_i \in S$ with $t_i \rightarrow t_\infty$. By definition, for each t_i , there exist $u_i = u_{t_i}$, $\sigma_i = \sigma_{t_i}$ and $y_i = y_{t_i}$, and if we set $x_i = x'(\sigma_{t_i}) + Q_{\sigma_{t_i}} u_i$, then $g_i y_i = x_i$ holds. Then obviously $y_i = x' + t_i b$ converge to $y_\infty = x' + t_\infty b$ in C^∞ . Since Γ_θ is compact, σ_i converge to some σ_∞ . Then the spliced configurations $x'(\sigma_{t_i})$ converge to $x'(\sigma_{t_\infty})$ in C^∞ . By the uniform bound $\|u_i\|_U < \nu$, taking a subsequence, we have a weak limit u_∞ so that $u_i \rightarrow u_\infty$ in $U = L^{p, w_T}$. Then x_i converge weakly in L_1^{p, w_T} , and we may assume u_i converge weakly in L_2^{p, w_T} and strongly in L_1^{p, w_T} . Now we would like to see that u_i converge to u_∞ strongly. By (A.14),

$$(A.18) \quad \|u_i - u_j\|_U \leq \|\Theta(g_i y_i) - \Theta(g_j y_j)\|_U + \|\Theta(x'(\sigma_i)) - \Theta(x'(\sigma_j))\|_U + \|n(Q_{\sigma_i} u_i) - n(Q_{\sigma_j} u_j)\|_U.$$

If $i, j \rightarrow \infty$, then the second term of the right hand side $\|\Theta(x'(\sigma_i)) - \Theta(x'(\sigma_j))\|_U$ tends to 0, because $x'(\sigma_i)$ converge in C^∞ . The first term is estimated, for instance, as

$$\|\Theta(g_i y_i) - \Theta(g_j y_j)\|_U \leq \|g_i - g_j\|_{L^{p, w_T}} \cdot \|\Theta(y_j)\|_{C^0} + \|g_i\|_{L^{p, w_T}} \cdot \|\Theta(y_i) - \Theta(y_j)\|_{C^0},$$

where the right hand side tends to 0 if $i, j \rightarrow \infty$. For the third term,

$$\begin{aligned} \|n(Q_{\sigma_i} u_i) - n(Q_{\sigma_j} u_j)\|_U &\leq M \|Q_{\sigma_i} u_i - Q_{\sigma_j} u_j\|_U (\|Q_{\sigma_i} u_i\|_U + \|Q_{\sigma_j} u_j\|_U) \\ &\leq M (\|Q_{\sigma_i} - Q_{\sigma_j}\|_{\text{OP}} \|u_i\|_U + C_{\sigma_j} \|u_i - u_j\|_U) (C_{\sigma_i} \|u_i\|_U + C_{\sigma_j} \|u_j\|_U) \\ &\leq M (\|Q_{\sigma_i} - Q_{\sigma_j}\|_{\text{OP}} \|u_i\|_U + N \|u_i - u_j\|_U) \cdot N (\|u_i\|_U + \|u_j\|_U). \end{aligned}$$

where N is the constant in (A.15). If we assume $\|u_i\|_U < (4MN^2)^{-1}$, then, by rearranging (A.18), we can see that the sequence $\{u_i\}$ is a Cauchy sequence in U , and u_∞ is the strong limit.

Now we choose ν so that $\nu \leq \min\{(2N)^{-1}, (4MN^2)^{-1}\}$, and choose ε and T as in Lemma A.17. Then $\{u_i\}$ converge strongly to u_∞ , and by Lemma A.17, the limit u_∞ satisfies $\|u_\infty\|_U < \nu$. This means $t_\infty \in S$, and the closedness is proved.

A(iv). **Proof of Proposition A.11: Openness.** Let us prove the openness. To prove the openness, we use the implicit function theorem. Suppose $t_0 \in S$ with $0 \leq t_0 < 1$ so that there exist g_0 , σ_0 and u_0 so that $g_0 y_{t_0} = x'_0 + Q_0 u_0$. To prove is $[t_0, t_0 + \epsilon) \subset S$ for small ϵ . In fact, we will prove any configuration z close to y_{t_0} is gauge equivalent to some $x'_v + Q_v(u_0 + w)$ for some $v \in \text{Lie } \Gamma_\theta$ and $w \in U$, where $x'_v = x'(\sigma_0, v)$ and $Q_v = Q_{(\sigma_0 \exp v)}$. Define a map

$$\mathcal{F}: \Omega^0(i\lambda) \times \text{Lie } \Gamma_\theta \times (\Omega^+(i\lambda) \oplus \Gamma(S^-)) \rightarrow \Omega^1(i\lambda) \oplus \Gamma(S^+)$$

by

$$\mathcal{F}(f, v, w) = \exp(f)(x'_v + Q_v(u_0 + w)) - (x'_0 + Q_0(u_0)).$$

We need to show that \mathcal{F} is surjective onto a neighborhood of 0. This follows from the implicit function theorem if the derivative $D\mathcal{F}$ of \mathcal{F} at $(0, 0, 0)$ is surjective. If $x'_0 = (A'_0, \Phi'_0)$, the derivative $D\mathcal{F}$ is

$$D\mathcal{F}_{(0,0,0)}(f, v, w) = \mathcal{I}_{\Phi'_0}(f) + \partial_v x'_v + \partial_v Q_v(u_0) + Q_0(w),$$

where $\mathcal{I}_{\Phi'_0}(f) = (-df, f\Phi'_0)$ and ∂_v means the derivative with respect to v .

More precisely, $\partial_v x'_v$ can be written as follows: For the connection part $A'(\sigma_0, v)$ of x'_v , set

$$j(v) = \left. \frac{\partial}{\partial s} A'(\sigma_0, sv) \right|_{s=0} \in \Omega^1(i\lambda).$$

Then $j(v) = d(\lambda_2 v) = -d(\lambda_1 v)$ on $[-1, 1] \times Y$, and $j(v) = 0$ outside of $[-1, 1] \times Y$. Now we have

$$\partial_v x'_v = (j(v), 0) \in \Omega^1(i\lambda) \oplus \Gamma(S^+).$$

The term $\partial_v Q_v$ will be discussed below.

In order to prove the surjectivity of $D\mathcal{F}$, we define a map

$$\mathcal{T}: \Omega^0(i\lambda) \times \text{Lie } \Gamma_\theta \times (\Omega^+(i\lambda) \oplus \Gamma(S^-)) \rightarrow \Omega^1(i\lambda) \oplus \Gamma(S^+)$$

by

$$\mathcal{T}(f, v, w) = \mathcal{I}_{\Phi'_0}(f) + (j(v), 0) + Q_0(w) = D\mathcal{F} - \partial_v Q_v(u_0).$$

Let B_1 be the completion of the domain of \mathcal{T} in the norm:

$$\|(f, v, w)\|_{B_1} = \|\mathcal{I}_{\Phi'_0}(f) + j(v)\|_V + \|w\|_U,$$

where $U = L^{p, w_T}$ and $V = L^{q, w_T}$. This is a norm by Lemma A.19 below. Let B_2 be the completion of the range in the norm:

$$\|(a, \phi)\|_{B_2} = \|(a, \phi)\|_V + \|\mathcal{D}_{x'_0}(a, \phi)\|_U.$$

Now, the fact that $\|\cdot\|_{B_1}$ is a norm follows from the following:

Lemma A.19. *If $H_{x_1}^0 = H_{x_2}^0 = 0$, then there exists a constant L independent of T such that, for any $f \in \Omega^0(i\lambda)$ and any $v \in \text{Lie } \Gamma_\theta$, we have*

$$\|f\|_{C^0} + |v| \leq L\|\mathcal{I}_{x'_0}(f) + j(v)\|_V.$$

Proof. Let $f_1 = f + (1 - \lambda_1)$ over $\text{supp}(\lambda_1) \subset X_1$, and $f_2 = f - (1 - \lambda_2)$ over $\text{supp}(\lambda_2) \subset X_2$. Then, for $i = 1, 2$,

$$df + j(v) = df_i$$

over X_i^T , and $f_1 - f_2 = v$ over $[-1, 1] \times Y$. Then each of $\|f\|_{C^0}$ and $|v|$ is bounded above by $\|f_1\|_{C^0} + \|f_2\|_{C^0}$. On the other hand, if $H_{x_i}^0 = 0$, then there exists a constant L_i for each i so that

$$\|f_i\|_{C^0} \leq L_i\|\mathcal{I}_{\Phi_i}(f_i)\|_{L^{q,w_i}},$$

since $q > 4$. Then we have

$$\|f_i\|_{C^0} \leq L_i\|\mathcal{I}_{\Phi'_i}(f_i)\|_{L^{q,w_i}} + \|\Phi_i - \Phi'_i\|_{L^{q,w_i}}\|f_i\|_{C^0}.$$

By the exponential decay, $\|\Phi_i - \Phi'_i\|_{L^{q,w_i}} = O(e^{-\delta T})$. So we can choose large T so that $\|\Phi_i - \Phi'_i\|_{L^{q,w_i}} < \frac{1}{2}$, say. Rearranging this, we obtain a bound for $\|f\|_{C^0}$ by $\|\mathcal{I}_{\Phi'_i}(f)\|_{L^{q,w_i}}$. Since $\mathcal{I}_{\Phi'_i}(f_i)$ is supported on $\text{supp } \lambda_i$, the L^{q,w_i} and L^{q,w_T} norms of it are uniformly equivalent, and the lemma is proved. \square

Thus \mathcal{T} is a bounded map from B_1 to B_2 . In fact, the following holds:

Lemma A.20. *There exists a constant K independent of T so that*

$$(A.21) \quad \|(f, v, w)\|_{B_1} \leq K\|\mathcal{T}(f, v, w)\|_{B_2}.$$

Proof. Let $\alpha = \mathcal{T}(f, v, w) = \mathcal{I}_{\Phi'_0}(f) + j(v) + Q_0(w)$. We consider $\mathcal{D}_{x'_0}\alpha$. By (4.15), we have

$$\mathcal{D}_{x'_0}\mathcal{I}_{\Phi'_0}(f) = (0, fD_{A'_0}\Phi'_0).$$

On the other hand,

$$\mathcal{D}_{x'_0}(j(v)) = (-d(j(v)), j(v)\Phi'_0) = 0,$$

because $\text{supp } j(v) \cap \text{supp } \Phi'_0 = \emptyset$. Thus we have

$$\mathcal{D}_{x'_0}\alpha = (0, fD_{A'_0}\Phi'_0) + w.$$

Since $\|f\|_{C^0} \leq L\|\mathcal{I}_{x'_0}(f) + j(v)\|_V$ by Lemma A.19, and $\|D_{A'_0}\Phi'_0\|_U = O(e^{-\delta T})$, we obtain

$$\begin{aligned} \|w\|_U &\leq \|\mathcal{D}_{x'_0}\alpha\|_U + \|f\|_{C^0}\|D_{A'_0}\Phi'_0\|_U \\ &\leq \|\mathcal{D}_{x'_0}\alpha\|_U + \text{const.}e^{-\delta T}\|\mathcal{I}_{x'_0}(f) + j(v)\|_V \\ &= \|\mathcal{D}_{x'_0}\alpha\|_U + \text{const.}e^{-\delta T}\|\alpha - Q_0(w)\|_V \\ &\leq \|\mathcal{D}_{x'_0}\alpha\|_U + \text{const.}e^{-\delta T}(\|\alpha\|_V + C\|w\|_U). \end{aligned}$$

Thus, when T is sufficiently large, we obtain a bound $\|w\|_U \leq K_1\|\alpha\|_{B_2}$ for some constant K_1 . Therefore we have

$$\|\mathcal{I}_{x'_0}(f) + j(v)\|_V = \|\alpha - Q_0(w)\|_V \leq (1 + CK_1)\|\alpha\|_{B_2}.$$

Combining the last two inequalities, we can find a constant K so that (A.21) holds. \square

Corollary A.22. *The kernel of \mathcal{T} is zero, and the image of \mathcal{T} is closed in B_2 .*

Now we use the index theorem to prove \mathcal{T} is the isomorphism.

Proposition A.23. *If $H_{x_i}^0 = H_{x_i}^1 = H_{x_i}^2 = 0$ for $i = 1, 2$, then the operator \mathcal{T} is an isomorphism from B_1 to B_2 with operator norm $\|\mathcal{T}^{-1}\|_{\text{OP}} \leq K$.*

Proof. The operator Q_0 is a pseudo-differential operator whose symbol is homotopic to $(D_{x'_0})^*(1 + (D_{x'_0})^*D_{x'_0})^{-1}$. Thus $\mathcal{I}_{\Phi'_0} \oplus Q_0$ is Fredholm, and the index is calculated as

$$\text{ind} [\mathcal{I}_{\Phi'_0} \oplus Q_0] = \text{ind} [(\mathcal{I}_{\Phi'_0})^* \oplus \mathcal{D}_{x'_0}]^* = -\text{ind} [(\mathcal{I}_{\Phi'_0})^* \oplus \mathcal{D}_{x'_0}].$$

Then

$$\begin{aligned} \text{ind } \mathcal{T} &= \dim \text{Lie } \Gamma_\theta - \text{ind} [(\mathcal{I}_{\Phi'_0})^* \oplus \mathcal{D}_{x'_0}] \\ &= \dim \text{Lie } \Gamma_\theta - \{\text{ind} [(\mathcal{I}_{\Phi_1})^* \oplus \mathcal{D}_{x_1}] + \text{ind} [(\mathcal{I}_{\Phi_2})^* \oplus \mathcal{D}_{x_2}] + \dim \text{Lie } \Gamma_\theta\} = 0. \end{aligned}$$

Now the proposition immediately follows from Corollary A.22. \square

Recall $D\mathcal{F} = \mathcal{T} + \partial Q_v(u_0)$, and we have seen that \mathcal{T} is an isomorphism from B_1 to B_2 which satisfies (A.21). If we see the operator norm of the map $v \mapsto \partial Q_v(u_0)$ is less than K^{-1} in (A.21), then \mathcal{F} is also invertible, and the proof of Proposition A.11 is completed.

Let us evaluate the norm of $\partial Q_v(u_0)$. Recall Q_v is constructed as

$$(A.24) \quad Q_v = Q'_v (\mathcal{D}_{x'_v} Q'_v)^{-1},$$

where Q'_v is the spliced operator which can be written as

$$Q'_v = Q'_{v,1} + Q'_{v,2},$$

with

$$Q'_{v,i} = h_i Q'_i h_i^{-1}, \quad \text{and} \quad Q'_i(u) = \gamma_i Q_i(\chi_i u),$$

where h_i are the gauge transformations in (5.3), and Q_i , γ_i and χ_i are defined around Proposition A.4. Then the differential of Q'_v with respect to v at $v = 0$ is given by

$$\partial_v Q'_v(u) = [(1 - \lambda_1)v_1, Q_1 u] + [(1 - \lambda_2)v_2, Q_2 u].$$

Then we have

$$\|\partial_v Q'_v(u)\|_V \leq \text{const.} |v| \|u\|_U.$$

Similarly, the differential $\partial_v (\mathcal{D}_{x'_v} Q'_v(u))$ is bounded as

$$\|\partial_v (\mathcal{D}_{x'_v} Q'_v(u))\|_V \leq \text{const.} |v| \|u\|_U.$$

By differentiating (A.24), we obtain

$$\partial_v Q_v = \{\partial Q'_v - Q_0 \partial (\mathcal{D}_{x'_v} Q'_v)\} (\mathcal{D}_{x'_0} Q'_0)^{-1}.$$

Hence we obtain the estimate

$$(A.25) \quad \|\partial_v Q_v(u)\|_V \leq \text{const.} |v| \|u\|_U.$$

Differentiating the identity $\mathcal{D}_{x'_v} Q_v = 1$, we have

$$\mathcal{D}_{x'_0} (\partial_v Q_v(u)) = -(\partial_v \mathcal{D}_{x'_v}) Q_0(u) = (0, -j(v)\phi),$$

where ϕ is the spinor component of $Q_0(u)$. Therefore we have the estimate

$$\|\mathcal{D}_{x'_0}(\partial_v Q_v(u))\|_U \leq \text{const.} \|j(v)\|_{L^4} \|Q_0(u)\|_V \leq \text{const.} |v| \|u\|_U,$$

because of the following facts:

- Hölder's inequality with (A.7) implies $\|ab\|_U \leq \|a\|_V \|b\|_{L^4}$, and
- we may assume the L^4 norm of $\nabla \lambda_i$ is independent of T , and therefore $\|j(v)\|_{L^4} \leq \text{const.} |v|$.

Summing up these, we obtain

$$\|\partial_v Q_v(u)\|_{B_2} \leq \text{const.} |v| \|u\|_U \leq \text{const.} \|(f, v, u)\|_{B_2} \|u\|_U.$$

Now if $\|u\|_U$ is small (i.e., ν is small), then $D\mathcal{F}$ is invertible and the proof of Proposition A.11 is completed.

A(v). The injectivity of the map I . Now, we prove that the map I is injective:

Proposition A.26 ([4], §7.2.6). *For monopoles x_i on X_i with $H_{x_i}^2 = 0$ ($i = 1, 2$), and for sufficiently small ε , the map I of (5.2) is injective.*

Proof. If $H_{x_i}^0 \neq 0$ for some i , then Gl is one point, and therefore I is obviously injective. Suppose $H_{x_i}^0 = 0$ for $i = 1, 2$. For the fixed identification σ_0 and any v , suppose the following:

- $I(\sigma_0)$ is represented by $x'(\sigma_0) + y_0$.
- $I(\sigma_0 \exp v)$ is represented by $x'(\sigma_0, v) + y_v$.
- $x'(\sigma_0) + y_0$ and $x'(\sigma_0, v) + y_v$ are gauge equivalent.
- $x'(\sigma_0)$ and $x'(\sigma_0, v)$ are not gauge equivalent.

First, we claim that, if $x'(\sigma_0) + y_0$ and $x'(\sigma_0, v) + y_v$ are gauge equivalent, then we may assume they are equivalent by a gauge transformation in the identity component. Recall that $\pi_0 \mathcal{G} \cong H^1(X; l) \cong \mathbb{Z}^{b_1(X; l)} \oplus \mathbb{Z}_2$. Let $\rho: \mathcal{G} \rightarrow \pi_0 \mathcal{G}$ be the projection. Let us consider the case when both of x_1 and x_2 are monopoles on twisted Spin^{c-} -structures. (The untwisted Spin^{c-} -cases are easier.) Let us write the connection terms of $x'(\sigma_0)$ and $x'(\sigma_0, v)$ as $A(\sigma_0)$ and $A(\sigma_0, v)$, and let a_0 and a_v be the 1-form components of y_0 and y_v . By Lemma 5.7, there exists $t \in H^1(X; l)$ which is represented by a gauge transformation \check{g} such that $A(\sigma_0, \pi i) = \check{g} A(\sigma_0)$. Hence, as de Rham classes, $n[t] = [A(\sigma_0, n\pi i) - A(\sigma_0)]$ for $n \in \mathbb{Z}$. Suppose $x'(\sigma_0) + y_0 = g(x'(\sigma_0, v) + y_v)$ for some $g \in \mathcal{G}$. Since the de Rham classes of a_0 and a_v are very small for large T , we see that $[A(\sigma_0) + a_0 - (A(\sigma_0, v) + a_v)]$ should be $n[t]$ for some $n \in \mathbb{Z}$, and therefore $\rho(g)$ is in $\mathbb{Z}\langle t \rangle \oplus \mathbb{Z}_2$. Then by replacing $x'(\sigma_0, v)$ by $(\pm 1) \cdot \check{g}^{-n} \cdot x'(\sigma_0, v)$, we may assume $x'(\sigma_0) + y_0$ and $x'(\sigma_0, v) + y_v$ are gauge equivalent by a gauge transformation of the form $g = \exp(\chi)$ for some $\chi \in \Omega^0(i\lambda)$. By restricting on X_i^{2T} , we obtain gauge transformations g_i over X_i^{2T} so that $x'(\sigma_0) + y_0 = g_i(x'(\sigma_0, v) + y_v)$. Then, for the connection parts, we have

$$A'_i(\sigma_0) + a_0 = g_i(A'_i(\sigma_0, v) + y_v) = g_i h_i(A'_i(\sigma_0) + a_v),$$

where h_i are the gauge transformations in (5.3). Set $g'_i = g_i h_i$. We may assume $g'_i = \exp(\chi_i)$ for some χ_i . Then we have $-2d\chi_i = a_0 - a_v$, and therefore

$$\|\chi_i\|_{C^0} \leq \text{const.} \|d\chi_i\|_{L^{q,w_i}(X_i^{2T})} \leq \text{const.} \|a_0 - a_v\|_{L^{q,w_i}(X_i^{2T})}.$$

On the overlapping region, the compatibility condition for g_i implies $|\chi_1 - \chi_2| = |v|$. Thus we have

$$(A.27) \quad |v| \leq \text{const.} \|a_0 - a_v\|_V.$$

On the other hand, y_v is given as $y_v = Q_v(u_v)$ for a u_v such that $u_v + n(Q_v(u_v)) = -\Theta(x'(\sigma_o, v))$. Since $\Theta(x'(\sigma_o, v))$ is supported on the region where $h_i = 1$, the v -derivative of u_v is given by $\partial_v u_v = -\partial_v(n(Q_v(u_v)))$. By calculating the derivative (by using (A.9)), we have

$$\|\partial_v(n(Q_v(u_v)))\|_U \leq \text{const.} \|\partial_v(Q_v(u_v))\|_V \cdot \|Q(u)\|_V \leq \text{const.} \|(\partial_v Q_v)(u) + Q(\partial_v u_v)\|_V \cdot \|u\|_U$$

Since $\|u\|_U \leq \text{const.} \varepsilon$, the estimate (A.25) implies

$$\|\partial_v u_v\|_U \leq \text{const.} (|v| \varepsilon + \|\partial_v u_v\|_U) \varepsilon$$

Rearranging this, we have $\|\partial_v u_v\|_U \leq \text{const.} |v| \varepsilon^2$, and hence

$$|v| \leq \text{const.} \|a_0 - a_v\|_V \leq \text{const.} |v| \varepsilon^2.$$

Thus for small ε , we obtain $v = 0$. □

A(vi). **Gluing map** Ξ . Suppose $\mathcal{M}_i = \mathcal{M}(X_i)$ ($i = 1, 2$) are regular. Since \mathcal{M}_i are compact, we can define for sufficiently large T the global gluing map,

$$(A.28) \quad \Xi: \tilde{\mathcal{M}}_1 \times_{\Gamma_\theta} \tilde{\mathcal{M}}_2 \rightarrow \mathcal{M}(X^{\#T}).$$

First we will prove the map Ξ is diffeomorphism when \mathcal{M}_1 and \mathcal{M}_2 are acyclic with the assumptions of Theorem 5.4:

Theorem A.29. *Suppose the assumptions of Theorem 5.4 are satisfied. Suppose further that \mathcal{M}_1 and \mathcal{M}_2 are acyclic. Then, for a large T , the gluing map (A.28) is a diffeomorphism, and $\mathcal{M}(X^{\#T})$ is regular.*

For the proof, we need some more things. Let ε and $T(\varepsilon)$ be the constants in Proposition A.11, and take $T > T(\varepsilon)$. For τ such that $T > \tau > T(\varepsilon)$, let $K_1^\tau = X_1^\tau$, $K_2^\tau = X_2^\tau$ and $K^\tau = K_1^\tau \cup K_2^\tau$. We can assume K^τ as a submanifold of $X^{\#T}$. So by restricting to K^τ , we can compare configurations on the different manifolds $X_1 \cup X_2$ and $X^{\#T}$. Let $\mathcal{B}(K^\tau)$ be the space of the configurations modulo gauge over K^τ . We may identify $\mathcal{B}(K^\tau) = \mathcal{B}(K_1^\tau) \times \mathcal{B}(K_2^\tau)$. For $a = [x_1] \times [x_2]$ and $b = [y_1] \times [y_2]$ in $\mathcal{B}(K^\tau)$, we define the metric

$$d_{K^\tau}(a, b) = \inf_{g_1 \in \mathcal{G}(K_1^\tau)} \|g_1 x_1 - y_1\|_V + \inf_{g_2 \in \mathcal{G}(K_2^\tau)} \|g_2 x_2 - y_2\|_V.$$

For monopoles x_i ($i = 1, 2$) on X_i , let x'_i be the flattened configuration, and $J: \text{Gl} \rightarrow \mathcal{B}(X^{\#T})$ the map splicing x'_1 and x'_2 with a gluing parameter σ . If w is a monopole on $X^{\#T}$, then there exists a constant C such that,

$$d_{K^\tau}([w]_{K^\tau}, [x'_1] \times [x'_2]_{K^\tau}) < Cd([w], J(\text{Gl})).$$

Conversely, we have

Proposition A.30. *There exists a constant τ with $\tau > T(\varepsilon)$ such that if*

$$d_{K^\tau}([w]|_{K^\tau}, [x'_1] \times [x'_2]|_{K^\tau}) < \frac{\varepsilon}{2},$$

then

$$d([w], J(\text{Gl})) < \varepsilon.$$

Proof. Let us consider the disjoint union $K^T = K_1^T \cup K_2^T$, where $K_i^T = X_i^T$. Note that $X^{\#T}$ is made by gluing K_1^T and K_2^T . For a monopole x on $X^{\#T}$, let us consider the restriction $[x]|_{K^T} = [x]|_{K_1^T} \times [x]|_{K_2^T} \in \mathcal{B}(K_1^T) \times \mathcal{B}(K_2^T)$. Then $d_{K^T}([x]|_{K^T}, [x'_1] \times [x'_2]|_{K^T}) < \varepsilon$ implies $d([x], J(\text{Gl})) < \varepsilon$. Let $K^{T-\tau}$ be the disjoint union of $(X_1^T \setminus X_1^\tau)$ and $(X_2^T \setminus X_2^\tau)$. The exponential decay estimate implies that there exists a constant C such that, for every monopole w on $X^{\#T}$ and every monopoles x_1 and x_2 on X_1 and X_2 ,

$$d_{K^{T-\tau}}([w], [x'_1] \times [x'_2]) < Ce^{-\delta\tau}.$$

Hence if τ is large enough, then, say, $d_{K^{T-\tau}}([w], [x'_1] \times [x'_2]) < \varepsilon/10$, and the proposition holds. \square

By Corollary A.12, we obtain the following:

Corollary A.31. *For τ in Proposition A.30, if w is a monopole on $X^{\#T}$ with $d_{K^\tau}([w]|_{K^\tau}, [x'_1] \times [x'_2]|_{K^\tau}) < \varepsilon/2$ as above, then $[w]$ is in the image $I(\text{Gl})$.*

In order to define the inverse of the gluing map, we need to make monopoles on X_i from a monopole on $X^{\#T}$.

Suppose x is a monopole on $X^{\#T}$ with $H_x^2 = 0$. Let us consider the configuration x' obtained by making x flattened on the neck. More precisely, using the function γ in §A(i), we define the function $\bar{\gamma}$ by

$$\bar{\gamma}(t) = \begin{cases} \gamma(-t-3), & t \geq 0, \\ \gamma(t+3), & t < 0, \end{cases}$$

and let

$$x' = \bar{\gamma}x + (1 - \bar{\gamma})(\theta, 0).$$

For each i , restricting x' to X_i^T , assuming $X_i^T \subset X_i$ and extending x' over X_i obviously, we obtain an approximate monopole x'_i on each X_i . Taking a large T and arguing as in §A(i), we can construct, for each $i = 1, 2$, a genuine monopole y_i on X_i which is close to x'_i . To do this, first we need to construct a right inverse Q_i for the operator $\mathcal{D}_{x'_i}$ for each i . The operator Q_1 , say, is constructed by splicing the right inverse $Q_{x'}$ for $\mathcal{D}_{x'}$ over $X^{\#T}$ with the right inverse for the operator $\mathcal{D}_{(\theta, 0)}$ over the cylinder $(-2T, \infty) \times Y$ as in §A(i). Then, by the contraction mapping principle, we can find a genuine monopole y_i near x'_i for each i . Taking a large T , we may assume

$$d_{K^\tau}([x]|_{K^\tau}, [y'_1] \times [y'_2]|_{K^\tau}) < \frac{\varepsilon}{2}.$$

So the monopole class $[x]$ is in the image $I(\text{Gl})$ for gluing y_1 and y_2 . By Proposition A.26, we find the inverse image of $[x]$ for the gluing map Ξ , and we can see that Ξ is a diffeomorphism in the acyclic case. Thus Theorem A.29 for the acyclic cases is proved.

In order to generalize Theorem A.29 to the non-acyclic cases, we need one more ingredient. If $\dim \mathcal{M}_1$ or $\dim \mathcal{M}_2$ is positive, cutting down the moduli space reduces the argument to the 0-dimensional case. For given points $w_i = [x_i] \in \mathcal{M}_i$ ($i = 1, 2$), choose local coordinates,

$$\varphi_i: U_i \subset \mathcal{M}_i \rightarrow \mathbb{R}^{d_i}.$$

By §10.4 of [7], we can embed $U_1 \times U_2$ into $\mathcal{B}(K^\tau)$ for a large τ via the restriction map $r: \mathcal{B}(X_1) \times \mathcal{B}(X_2) \rightarrow \mathcal{B}(K^\tau)$, and find an open neighborhood V of $r(U_1 \times U_2)$ and a map

$$q: V \rightarrow U_1 \times U_2,$$

satisfying $q \circ r$ is the identity on $U_1 \times U_2$. Let $r': \mathcal{B}(X^{\#T}) \rightarrow \mathcal{B}(K^\tau)$ be the restriction map, and $\mathcal{U}(\varepsilon)$ be an open set of (A.10) for the configurations splicing x_1 and x_2 . We can arrange so that $r'(\mathcal{U}(\varepsilon)) \subset V$. Let us define Υ by the composite map as

$$\Upsilon = (\varphi_1 \times \varphi_2) \circ q \circ r': \mathcal{U}(\varepsilon) \rightarrow \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}.$$

Then, the argument in the previous subsections work for the cut-down moduli space $\Upsilon^{-1}(z_1, z_2) \cap \mathcal{U}(\varepsilon) \cap \mathcal{M}(X^{\#T})$ for $(z_1, z_2) \in \varphi_1(U_1) \times \varphi_2(U_2)$. In particular, Corollary A.12 is modified to the following:

Proposition A.32. *For each $i = 1, 2$, let x_i be a monopole on X_i with $[x_i] \in U_i$, and set $z_i = \varphi_i([x_i])$. If $H_{x_i}^0 = H_{x_i}^2 = 0$ for $i = 1, 2$, then for small enough ε there exists $T(\varepsilon)$ so that for $T > T(\varepsilon)$ the intersection $\Upsilon^{-1}(z_1, z_2) \cap \mathcal{U}(\varepsilon) \cap \mathcal{M}(X^{\#T})$ is equal to the image of $I: \Gamma_\theta \rightarrow \mathcal{M}$.*

With this understood, we have a smooth family of diffeomorphisms parameterized by $(z_1, z_2) \in U_1 \times U_2$,

$$I_{(z_1, z_2)}: \Gamma_\theta / (\Gamma_{x_1} \times \Gamma_{x_2}) \rightarrow \Upsilon^{-1}(z_1, z_2) \cap \mathcal{U}(\varepsilon) \cap \mathcal{M}(X^{\#T}).$$

By using this, Theorem A.29 is generalized to Theorem 5.4 for $\mathcal{M}(X_i)$ with vanishing 0th and 2nd cohomology. Now the case when $\mathcal{M}(X_2)$ contains $U(1)$ -reducibles (especially Corollary 5.5) can be proved substantially more easily by modifying the arguments in the preceding subsections.

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